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Twin paradox in de Sitter spacetime

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Abstract

The 'twin paradox' of special relativity offers the possibility of making interstellar flights within a lifetime. For very long journeys with velocities close to the speed of light, however, we have to take into account the expansion of the universe. Inspired by the work of Rindler on hyperbolic motion in curved spacetime, we study the worldline of a uniformly accelerated observer in de Sitter spacetime and the communication between the travelling observer and an observer at rest. This paper is intended to give graduate students who are familiar with special relativity and have some basic experience of general relativity a deeper insight into accelerated motion in general relativity, into the relationship between the proper times of different observers and the propagation of light signals between them, and into the use of compactification to describe the global structure of a relativistic model.

1. Introduction

The 'twin paradox' is one of the most discussed problems in special relativity [1, 2]. The term 'paradox', however, is quite misleading, as all ramifications of the situation considered are well understood nowadays. Nevertheless, we use this term to sketch the basic situation of this paper. While one of the twins, we will call him Eric, stays on Earth, the other twin, Tina, undertakes a relativistic trip to some distant location in the universe and immediately returns home. In a previous work, Müller *et al* [3] discussed this situation in flat Minkowski space for a uniformly accelerating twin. There, the journey is separated into four stages of equal time lengths. In the first two stages, Tina accelerates and decelerates again until she reaches her destination. In the last two stages, she returns in exactly the same manner. As long as Tina visits only a nearby star, the special relativistic treatment is a sufficient approximation. But, if a trip to some far distant galaxies is considered, the expansion of spacetime must be taken into account.

The aim of this paper is to discuss the twin paradox situation for a uniformly accelerating twin in de Sitter spacetime. Although our own universe is not a de Sitter spacetime, nevertheless this will reveal the new aspects of the twin paradox that come about in an expanding universe. As pointed out by Rindler [4] it has the didactic merit that in it all the necessary integrations can be performed in terms of elementary functions. We extend the calculations by Rindler to incorporate not only the acceleration in one direction but also the complete journey. For very long acceleration times or very large Hubble constants, the temporal course of the journey will then differ significantly from the flat Minkowskian situation. Besides the worldline of the twin, we also study the influence of the expanding universe and the accelerated motion on the communication between both twins.

The discussion in this paper is addressed to graduate students who have finished a course in special relativity and have some basic knowledge of general relativity and cosmology. It might serve as a bridge to understand the transition between flat Minkowskian space and an empty expanding spacetime that is solely affected by the cosmological constant.

A detailed discussion of the geometric structure of de Sitter space [5, 6] can be found in Rindler [1]. The more advanced reader might be interested in the explanations by Hawking and Ellis [7], Schmidt [8] or Spradlin *et al* [9].

There are also some recent articles concerning accelerated motion in de Sitter space. Bičák and Krtouš [10], for example, study accelerated sources in de Sitter space and give a list of several coordinate representations. Podolský and Griffiths [11] discuss uniformly accelerating black holes in a de Sitter universe. From another point of view, Doughty [12] discusses the necessary acceleration in Schwarzschild spacetime to keep a fixed distance from the black hole horizon. A brief history of the cosmological constant is given by Harvey and Schucking [13]. Some recent publications on uniform acceleration within special relativity are, for example, by Semay [14], who presents a uniformly accelerated observer within a Penrose–Carter diagram, or Flores [15], who is concerned with the communication between accelerated observers. Heyl [16] and Kwan [17] study the ramifications of space travel in accelerating spacetimes. Zimmermann [18] discusses the problem of overtaking an object receding due to cosmic expansion.

The structure of this paper is as follows. In section 2, we briefly summarize the twin paradox journey in flat Minkowski space. In section 3, we introduce the form of the de Sitter metric we use in this work and recapitulate some properties needed for our discussion. In section 4, we derive the worldline of a twin paradox journey in de Sitter spacetime and discuss the differences from the journey in Minkowski space. In section 5, we scrutinize the influence of the expansion on the communication between both twins.

In addition to this paper, we have created two Maple worksheets for the interested reader to study the situations considered in this work in even more detail. These worksheets can be downloaded from http://www.itp1.uni-stuttgart.de/arbeitsgruppen/wunner/TwindeSitter/ .

2. Twin paradox in flat space

In flat Minkowski space, the twin paradox journey works as follows. While Eric stays on Earth, Tina starts her trip with zero velocity and moves with uniform acceleration α. After a proper time τ_1, with respect to her own clock, she decelerates again with α until she comes to rest at a proper time $2\tau_1$. Then, she immediately returns to Earth using the same procedure and reaches her brother at $4\tau_1$. Here, and for the rest of this paper, we denote these four stages with the circled numbers ①–④. Because the accelerations in the stages ② and ③ point in the same direction, each worldline is composed of three branches. Branch (a) describes stage ①; stages ② and ③ are combined into branch (b), and stage ④ is represented by branch (c), see figure 1. Hence, Tina's worldline with respect to Minkowski spacetime, $\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2)$, reads $(t = t_{\text{flat}}(\tau), r = r_{\text{flat}}(\tau), \vartheta = \pi/2, \varphi = 0)$

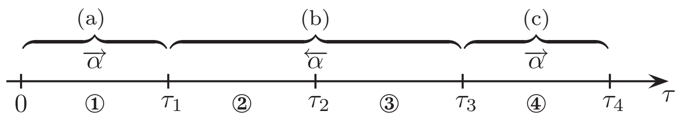

Figure 1. Branches, acceleration directions and proper times with respect to Tina. In Minkowski space, we have $\tau_n = n\tau_1$ for $n = \{2, 3, 4\}$.

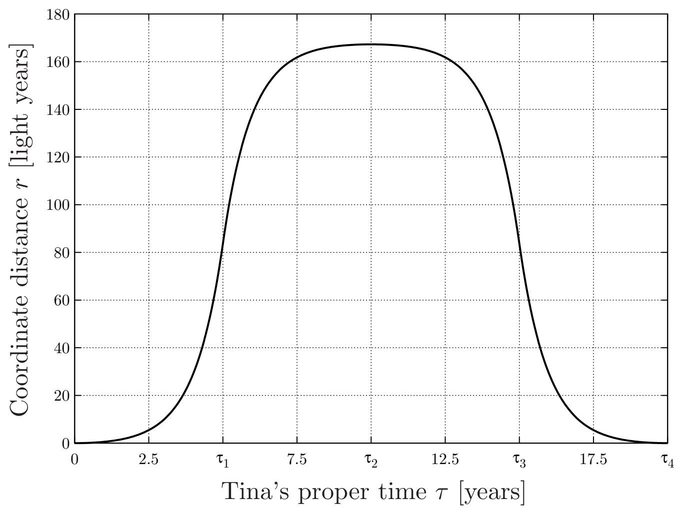

Figure 2. Radial coordinate distance r for a round trip with $4\tau_1 = 20\,\mathrm{y}$. Tina's maximum distance from Eric is $r_{\max} \approx 167.19\,\mathrm{ly}$.

with

$$t_{\text{flat}}(\tau) = \frac{c}{\alpha}\begin{cases} \sinh\left(\frac{\alpha}{c}\tau\right) & (1a) \\ \sinh\left[\frac{\alpha}{c}(\tau - 2\tau_1)\right] + 2\sinh\left(\frac{\alpha}{c}\tau_1\right) & (1b) \\ \sinh\left[\frac{\alpha}{c}(\tau - 4\tau_1)\right] + 4\sinh\left(\frac{\alpha}{c}\tau_1\right) & (1c) \end{cases}$$

and

$$r_{\text{flat}}(\tau) = \frac{c^2}{\alpha}\begin{cases} \cosh\left(\frac{\alpha}{c}\tau\right) - 1 & (2a) \\ 2\cosh\left(\frac{\alpha}{c}\tau_1\right) - \cosh\left[\frac{\alpha}{c}(\tau - 2\tau_1)\right] - 1 & (2b) \\ \cosh\left[\frac{\alpha}{c}(\tau - 4\tau_1)\right] - 1. & (2c) \end{cases}$$

A derivation of this worldline can be found in appendix A or in Müller *et al* [3]. As an example, figure 2 shows Tina's worldline with respect to her proper time τ. In each stage, she accelerates or decelerates with $\alpha = 9.81\,\mathrm{m\,s^{-2}} \approx 1.0326\,\mathrm{ly/y^2}$ for a time $\tau_1 = 5\,\mathrm{y}$. Thus, her journey lasts $4\tau_1 = 20\,\mathrm{y}$. On Earth, however, $t(4\tau_1) \approx 338.36\,\mathrm{y}$ pass by, as can easily be calculated from (1c). The maximum coordinate distance, $r_{\max}$, Tina can reach with this procedure, is given by (2b), setting $\tau = 2\tau_1$. In figure 3, the coordinate time $t(\tau)$ is shown

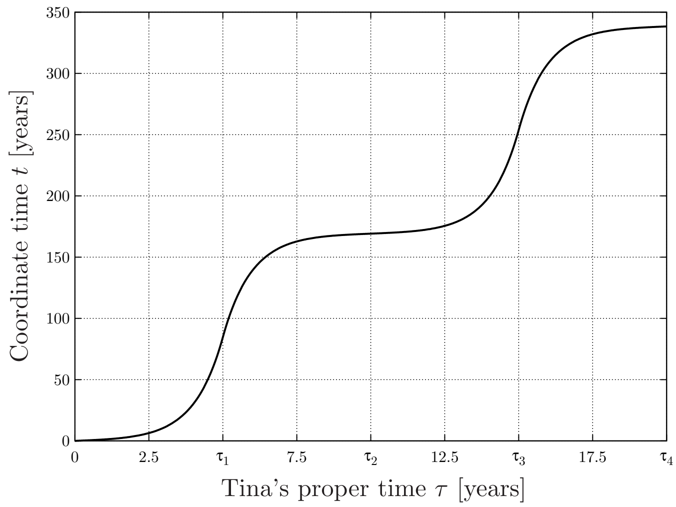

Figure 3. Coordinate time t for the round trip of figure 2.

for this trip. Note the pointwise symmetry of $t(\tau)$ around $2\tau_1$. This feature is lost in de Sitter space.

3. Basic properties of de Sitter space

As we know from current cosmological observations, our universe seems to have started in a big bang and has been expanding ever since. The most plausible assumptions for a cosmological model are that our universe is homogeneous and isotropic. This means that the three-dimensional space has maximum symmetry and thus has constant spatial curvature, although the curvature can be time dependent. In the context of general relativity, this cosmological model leads to the Fermi–Robertson–Walker (FRW) spacetimes. The time-dependent behaviour of the FRW spacetimes follows from the Einstein field equations. In the case of zero density, positive cosmological constant and zero curvature index, we obtain the de Sitter spacetime. In Lemaître–Robertson (LR) [19, 20] coordinates, the de Sitter metric is defined by the line element

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{e}^{2Ht}(\mathrm{d}r^2 + r^2\,\mathrm{d}\omega^2), \tag{3}$$

where $\mathrm{d}\omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2$ is the spherical surface element, c is the speed of light and H is the Hubble constant. Please note that equation (3) covers only a part of the whole de Sitter spacetime just as the Schwarzschild metric covers only a part of the Kruskal spacetime. But here we need only that part of the spacetime that is expressed by equation (3). Furthermore, even though this metric shows only spherical symmetry in space, the de Sitter spacetime is actually a four-dimensional space of constant curvature as can be verified by means of the Riemann tensor. For details, we refer the reader to the explanations by Rindler [1].

As the de Sitter spacetime is a solution of the Einstein field equations with zero density, it has little value for describing our own universe. But because of its simplicity and to follow Rindler's [4] calculations, we use it here to extend our 'twin paradox' calculations to an expanding universe model.

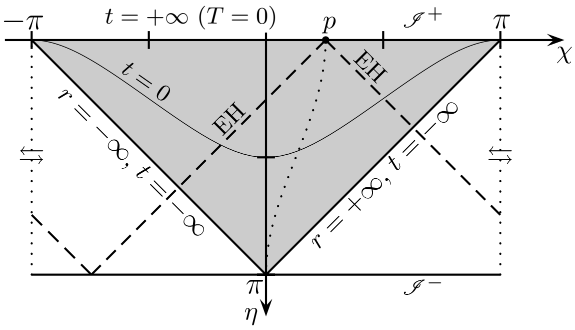

Figure 4. Penrose diagram of the de Sitter spacetime in the CE coordinates χ and η. The EH for a static observer $\mathcal{O}$ (dotted line) at p is indicated by dashed lines. Note that $t=-\infty$ corresponds to $T=+\infty$, and the point $t=r=0$ corresponds to $\chi=0$, $\eta=\pi$. The left–right arrows indicate that the 'lines' $\chi=-\pi$ and $\chi=\pi$ are identified. The grey-shaded triangle corresponds to the LR coordinate domain.

Furthermore, it can easily be shown that all infinitely expanding universes behave asymptotically like a de Sitter universe [1]. For a compact illustration of the worldlines of accelerating observers in de Sitter spacetime, we make use of an idea by Penrose and Carter, see also Semay [14], to compress a spacetime into a finite domain while preserving its causal structure. In the present case, this can be realized by the coordinate transformation $(t,r)\mapsto(\eta,\chi)$ with

$$\eta=\arctan\frac{2T\kappa}{\kappa^2-T^2+r^2},\qquad \chi=\arctan\frac{2r\kappa}{\kappa^2+T^2-r^2},\tag{4}$$

where $T=\kappa\exp(-ct/\kappa)$ and $\kappa=c/H$. The resulting line element in these so-called conformal Einstein (CE) coordinates [10] reads

$$\mathrm{d}s^2=\frac{\kappa^2}{\sin^2\eta}[-\mathrm{d}\eta^2+\mathrm{d}\chi^2+\sin^2\chi\,\mathrm{d}\omega^2].\tag{5}$$

Here, the coordinates η and χ are restricted to $\eta\in(0,\pi)$ and $\chi\in[-\pi,\pi]$, where $\chi=-\pi$ and $\chi=\pi$ are identified. If $\kappa^2-T^2+r^2<0$, we have to map $\eta\to\eta+\pi$. On the other hand, if $\kappa^2+T^2-r^2<0$, we have to consider the sign of r. If $r>0$, then $\chi\to\chi+\pi$; otherwise $\chi\to\chi-\pi$. Altogether, we have mapped an infinite spacetime onto a finite domain.

The advantage of the CE coordinates is that in the compact illustration, which is also called a Penrose diagram, see figure 4, radial light rays ($\mathrm{d}\vartheta=\mathrm{d}\varphi=0$) are represented by straight lines with $\pm45°$ slope. All past-directed light rays end at $\mathscr{I}^-$ $(\eta=\pi)$, pronounced 'scri minus', whereas all future-directed ones end at $\mathscr{I}^+$ $(\eta=0)$. All worldlines of particles, and of the travelling twin Tina in particular, are lines within the triangle bordered by the future-directed light ray starting at $r=0, t=0$ and the $\eta=0$ and $\chi=0$ coordinate lines. Hence, we can compare different situations very clearly using only a single diagram, which would be very difficult using LE coordinates. That is significant especially for $t\to\infty$. In contrast to the LR coordinates, the CE coordinates cover the whole spacetime. But, for our purpose, the LR coordinate domain is adequate. The hypersurface $t=0$ in CE coordinates is independent of κ:

$$\eta(\chi)=\arctan[\xi_+(\chi),\xi_-(\chi)]\tag{6}$$

with $\xi_\pm(\chi)=\frac{1}{2}(\mp\cos\chi+\sqrt{2-\cos^2\chi})$. The worldline of a static observer $\mathcal{O}$ with respect to the LR coordinates, $r=\text{const}$, is represented by the dotted line. The backward light cone

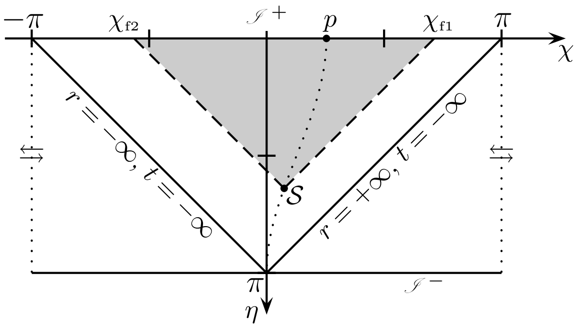

Figure 5. The dashed lines represent the forward light cone of $\mathcal{S}$ with coordinates ($t_{\mathcal{S}} = -0.5 \times 10^{-4}$, $r_{\mathcal{S}} \approx 0.423$) or ($\eta_{\mathcal{S}} \approx 1.482$, $\chi_{\mathcal{S}} \approx 0.435$), respectively, where we have chosen $\kappa = 1$. As in figure 4, the dotted line is the worldline of the static observer $\mathcal{O}$.

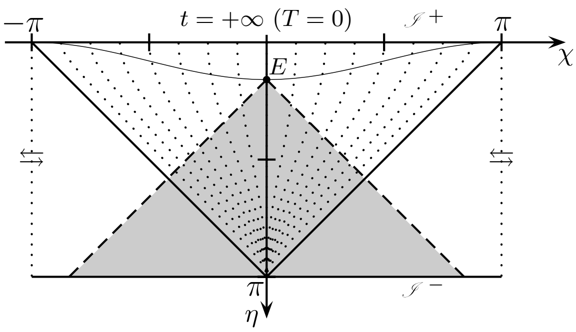

Figure 6. The dashed lines represent Eric's (E) backward light cone at the current observation time $\eta_{\mathrm{E}} = 0.5$ or $t_{\mathrm{E}} \approx 1.365$, respectively. The dotted lines represent static observers, $r = \mathrm{const}$, with respect to LR coordinates. The thin solid line indicates the hypersurface $t = t_{\mathrm{E}}$.

of $\mathcal{O}$'s 'final' point p defines his event horizon (EH). All events that lie beyond this horizon cannot influence the point p.

For our twin paradox journey, we are interested in the radial domain that can be reached by Tina when she starts at point $\mathcal{S}$ with coordinates $(t_{\mathcal{S}}, r_{\mathcal{S}})$ or $(\eta_{\mathcal{S}}, \chi_{\mathcal{S}})$, respectively. The radial domain follows from the forward light cone of $\mathcal{S}$, see figure 5. The critical points, where the forward light cone intersects $\eta = 0$, read $\chi_{\mathrm{f1}} = \chi_{\mathcal{S}} + \eta_{\mathcal{S}}$ and $\chi_{\mathrm{f2}} = \chi_{\mathcal{S}} - \eta_{\mathcal{S}}$. The corresponding LR radial coordinates are given by

$$r_{\mathrm{f1}} = \frac{\kappa \sin(\chi_{\mathcal{S}} + \eta_{\mathcal{S}})}{\cos(\chi_{\mathcal{S}} + \eta_{\mathcal{S}}) + 1} \quad \text{and} \quad r_{\mathrm{f2}} = \frac{\kappa \sin(\chi_{\mathcal{S}} - \eta_{\mathcal{S}})}{\cos(\chi_{\mathcal{S}} - \eta_{\mathcal{S}}) + 1}. \tag{7}$$

If $\chi_{\mathcal{S}} = 0$, (7) can be simplified to

$$r_{\mathrm{f1,f2}} = \pm\kappa\, \mathrm{e}^{-ct_{\mathcal{S}}/\kappa}. \tag{8}$$

Hence, if Tina starts at $\mathcal{S}$ with arbitrarily large acceleration, she can only move within the grey-shaded region. Eric's worldline is represented either by $(t, r_{\mathrm{E}} = 0)$ or by $(\eta, \chi_{\mathrm{E}} = 0)$, where $\eta = \arctan\{2\exp(-ct/\kappa)/[1 - \exp(-2ct/\kappa)]\}$. He can only observe events that lie inside his backward light cone (grey-shaded region in figure 6). The EH of Eric in CE coordinates is defined by $\eta = |\chi|$, when he is located at $(\eta_{\mathrm{E}} = 0, \chi_{\mathrm{E}} = 0)$. In LR coordinates, the EH simplifies to $r = \kappa \exp(-ct/\kappa)$. Hence, all particles at rest at some coordinate $r \neq 0$ eventually cross this EH. At time $t = 0$, particles at

$$r_{\mathrm{eh}} = \frac{c}{H} \tag{9}$$

cross Eric's EH. This value is particularly important in our discussion as we assume that Tina's journey starts at $t = 0$. We will show that the generalization to other times is trivial and provides no further insights.

In an expanding spacetime, the difference of coordinates of two points has little meaning as a measure of distance. Therefore, we follow Rindler [4] and introduce the proper radial distance

$$l = \mathrm{e}^{Ht} r \tag{10}$$

between Eric ($r_{\mathrm{E}} = 0$) and a point Q located at the coordinate distance r. The proper distance can be interpreted as the result of a distance measurement, where infinitely many stationary observers between Eric and Q sum up the distances they measure between each other at a given coordinate time [1]. Obviously, if $t = 0$, proper distance equals coordinate distance.

4. Twin paradox in de Sitter space

4.1. Derivation of the worldline

In general relativity, the worldline $x^{\mu}(\tau)$ of an individual moving with a constant proper acceleration α with respect to its local reference frame follows from

$$g_{\mu\nu} a^{\mu} a^{\nu} = \alpha^2 \tag{11}$$

with the four-acceleration

$$a^{\mu} = \frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2} + \Gamma^{\mu}_{\alpha\beta} \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\beta}}{\mathrm{d}\tau}. \tag{12}$$

Here, τ is the proper time of the individual and $\Gamma^{\mu}_{\alpha\beta}$ are the Christoffel symbols of the corresponding spacetime. In addition, the constraint equation $g_{\mu\nu} u^{\mu} u^{\nu} = -c^2$ for the four-velocity $u^{\mu} = \mathrm{d}x^{\mu}/\mathrm{d}\tau$ must be satisfied.

For the de Sitter space, the worldline of an individual who accelerates away from Earth for all times was first derived by Rindler [4]. To study a round trip as in Minkowski space, (11) has to be solved for all three branches of this journey separately. At the beginning, Tina starts from Earth with zero velocity and moves with constant acceleration α for some time τ_1. Thus, at Tina's proper time $\tau = 0$, we have

$$t = 0, \quad r = 0, \quad t' = 1, \quad r' = 0. \tag{13}$$

Note that here and in the following, a prime denotes differentiation with respect to τ.

Contrary to the situation in flat space, we allow different durations for the four stages of the journey for reasons we will present later. Taking into account that $t'(\tau)$ is continuous everywhere, we obtain

$$t'(\tau) = q \begin{cases} \dfrac{S\,\mathrm{e}^{2q\tau} + D}{\Psi(\tau)} & (14a) \\[2ex] \dfrac{\frac{D^2}{S} A^2_{\tau_1}\,\mathrm{e}^{2q\tau} + D}{\Psi_A(\tau)} & (14b) \\[2ex] \dfrac{\frac{D^2}{S} B^2_{\tau_3}\,\mathrm{e}^{2q\tau} + D}{\Psi_B(\tau)}, & (14c) \end{cases}$$

where

$$q = \sqrt{\left(\frac{\alpha}{c}\right)^2 + H^2}, \quad S = q + H, \quad D = q - H \tag{15}$$

and

$$\Psi(\tau) = HS\,\mathrm{e}^{2q\tau} + 2SD\,\mathrm{e}^{q\tau} - HD, \tag{16a}$$

$$A_{\tau_1} = \frac{\Omega_{\tau_1} S - \frac{\alpha}{c}[\Omega_{\tau_1} - 2\Psi(\tau_1)]}{\Omega_{\tau_1} D + \frac{\alpha}{c}[\Omega_{\tau_1} - 2\Psi(\tau_1)]}\mathrm{e}^{-q\tau_1}, \tag{16b}$$

$$B_{\tau_3} = \frac{\Omega_{\tau_3} S + \frac{\alpha}{c}[\Omega_{\tau_3} - 2\Psi_A(\tau_3)]}{\Omega_{\tau_3} D - \frac{\alpha}{c}[\Omega_{\tau_3} - 2\Psi_A(\tau_3)]}\mathrm{e}^{-q\tau_3}, \tag{16c}$$

$$\Omega_{\tau_1} = S\left(S + \frac{\alpha}{c}\right)\mathrm{e}^{2q\tau_1} + 2\left(SD - \frac{\alpha}{c}H\right)\mathrm{e}^{q\tau_1} + D\left(D - \frac{\alpha}{c}\right), \tag{16d}$$

$$\Omega_{\tau_3} = \frac{D^2}{S}A_{\tau_1}^2\left(S - \frac{\alpha}{c}\right)\mathrm{e}^{2q\tau_3} + 2\frac{D}{S}A_{\tau_1}\left(SD + \frac{\alpha}{c}H\right)\mathrm{e}^{q\tau_3} + D\left(D + \frac{\alpha}{c}\right), \tag{16e}$$

$$\Psi_A(\tau) = \frac{A_{\tau_1}^2 D^2}{S}H\,\mathrm{e}^{2q\tau} + 2A_{\tau_1}D^2\,\mathrm{e}^{q\tau} - HD, \tag{16f}$$

$$\Psi_B(\tau) = \frac{B_{\tau_3}^2 D^2}{S}H\,\mathrm{e}^{2q\tau} + 2B_{\tau_3}D^2\,\mathrm{e}^{q\tau} - HD. \tag{16g}$$

(Here, the branches (a)–(g) do not denote the different branches of the worldline!)

By definition, we have

$$t'(\tau) = \gamma(\tau) = \frac{1}{\sqrt{1 - \beta(\tau)^2}}, \tag{17}$$

with the Lorentz factor γ of special relativity and $\beta = v/c$, where v is Tina's velocity with respect to a local observer at rest at her current position. With (14) and (17), we obtain

$$\beta(\tau) = \frac{\alpha}{cq}\begin{cases} \dfrac{(\mathrm{e}^{q\tau} - 1)(S\,\mathrm{e}^{q\tau} + D)}{S\,\mathrm{e}^{2q\tau} + D} & (18a) \\[2ex] -\dfrac{\left(A_{\tau_1}\frac{D}{S}\,\mathrm{e}^{q\tau} - 1\right)\left(DA_{\tau_1}\,\mathrm{e}^{q\tau} + D\right)}{\frac{D^2 A_{\tau_1}^2}{S}\,\mathrm{e}^{2q\tau} + D} & (18b) \\[2ex] \dfrac{\left(B_{\tau_2}\frac{D}{S}\,\mathrm{e}^{q\tau} - 1\right)\left(DB_{\tau_2}\,\mathrm{e}^{q\tau} + D\right)}{\frac{D^2 B_{\tau_2}^2}{S}\,\mathrm{e}^{2q\tau} + D}, & (18c) \end{cases}$$

for her velocity during her trip. An important aspect in the following discussion is the maximum possible velocity on a journey. The maximum velocity is reached if Tina accelerates for all times. Using (14a) and (18a), we obtain

$$\gamma_\infty = \lim_{\tau\to\infty}\gamma(\tau) = \frac{q}{H}, \qquad \beta_\infty = \lim_{\tau\to\infty}\beta(\tau) = \frac{1}{q}\frac{\alpha}{c}. \tag{19}$$

Because $q > \alpha/c$, Tina's velocity asymptotically reaches some value smaller than 1, contrary to the situation in flat space, where $\beta \to 1$ for infinitely long trips. Integrating (14) over τ, and adjusting the constants of integration so that $t(\tau)$ is continuous and $t(0) = 0$, yields

$$t(\tau) = \frac{1}{H}\begin{cases} \ln\left[\dfrac{\Psi(\tau)}{2q^2\,\mathrm{e}^{q\tau}}\right] & (20a) \\[2ex] \ln\left[\dfrac{\Psi_A(\tau)}{2q^2\,\mathrm{e}^{q\tau}}\dfrac{\Psi(\tau_1)}{\Psi_A(\tau_1)}\right] & (20b) \\[2ex] \ln\left[\dfrac{\Psi_B(\tau)}{2q^2\,\mathrm{e}^{q\tau}}\dfrac{\Psi(\tau_1)}{\Psi_A(\tau_1)}\dfrac{\Psi_A(\tau_3)}{\Psi_B(\tau_3)}\right]. & (20c) \end{cases}$$

In the same manner, we obtain an expression for $r(\tau)$:

$$r(\tau) = \begin{cases} 2\alpha q \dfrac{H - S\,\mathrm{e}^{q\tau}}{HS\Psi(\tau)} + r_\infty & (21a) \\ -2\alpha q \dfrac{\Psi_A(\tau_1)}{\Psi(\tau_1)} \dfrac{H - DA_{\tau_1}\,\mathrm{e}^{q\tau}}{HDA_{\tau_1}\Psi_A(\tau)} + \mathcal{K}(\tau_1) + r_\infty & (21b) \\ 2\alpha q \dfrac{\Psi_A(\tau_1)}{\Psi(\tau_1)} \dfrac{\Psi_B(\tau_3)}{\Psi_A(\tau_3)} \dfrac{H - DB_{\tau_3}\,\mathrm{e}^{q\tau}}{HDB_{\tau_3}\Psi_B(\tau)} + \mathcal{H}(\tau_3) + \mathcal{K}(\tau_1) + r_\infty & (21c) \end{cases}$$

with

$$\mathcal{K}(\tau_1) = \frac{2\alpha q}{\Psi(\tau_1)} \frac{-2DSA_{\tau_1}\,\mathrm{e}^{q\tau_1} + H[S + A_{\tau_1}D]}{HDSA_{\tau_1}} \tag{22a}$$

$$\mathcal{H}(\tau_3) = \frac{2\alpha q}{\Psi(\tau_1)} \frac{\Psi_A(\tau_1)}{\Psi_A(\tau_3)} \frac{2DA_{\tau_1}B_{\tau_3}\,\mathrm{e}^{q\tau_3} - H[A_{\tau_1} + B_{\tau_3}]}{HDA_{\tau_1}B_{\tau_3}} \tag{22b}$$

and

$$r_\infty = \frac{\alpha}{HS} = \frac{c}{H} \frac{1}{\sqrt{1 + (Hc/\alpha)^2} + Hc/\alpha}. \tag{23}$$

Here, r_∞ is Rindler's 'α-horizon', which is the maximum coordinate distance that Tina can reach asymptotically when she accelerates away from Earth for all times starting at time $t_i = 0$ with zero initial velocity. Note that $r_\infty < r_{\mathrm{fl}} = c/H$, cf (7).

4.2. Analysis of the worldline

4.2.1. Comparison with flat space. The worldline $(t_{\mathrm{flat}}, r_{\mathrm{flat}})$ in flat space (see (1) and (2)) is a special case for $H \to 0$ of the more general worldline in de Sitter space. In order to show the similarity of the respective expressions, we rewrite (20a) and (21a) and obtain

$$r(\tau) = \frac{c^2}{\alpha} \frac{\mathrm{e}^{q\tau} + \mathrm{e}^{-q\tau} - 2}{2 + \zeta(\tau)}, \qquad t(\tau) = \frac{1}{H} \ln\left(\frac{2 + \zeta(\tau)}{2q^2\frac{c^2}{\alpha^2}}\right), \tag{24}$$

with

$$\zeta(\tau) = H\frac{c^2}{\alpha^2}(S\,\mathrm{e}^{q\tau} - D\,\mathrm{e}^{-q\tau}). \tag{25}$$

Because $\zeta(\tau) = 0$ and $q = \alpha/c$ for $H = 0$, we can directly see that $r(\tau) \to r_{\mathrm{flat}}(\tau)$ for $H \to 0$ and that $\zeta(\tau)$ characterizes the deviation of the $r(\tau)$-coordinate function from flat space for $H > 0$, except for the difference of q and α/c.

The situation is more difficult for $t(\tau)$. Using L'Hopital's law we can evaluate $t(\tau)$ in the limit $H \to 0$ and, indeed, obtain $t(\tau) \to t_{\mathrm{flat}}(\tau)$ in this case. The deviation from the flat space worldline is very small in the beginning, as H is very small. The current value of the Hubble constant is (see Hinshaw *et al* [21])

$$H_0 \approx (70.5 \pm 1.3)\frac{\mathrm{km}}{\mathrm{s}\cdot\mathrm{Mpc}} \approx (7.21 \pm 0.13) \times 10^{-11}\frac{\mathrm{ly}}{\mathrm{y}\cdot\mathrm{ly}}. \tag{26}$$

Thus, the effect of the expansion is negligible for trips within our galaxy, for example. However, the properties of the worldline for $\tau \to \infty$ change considerably. Most importantly, for $H > 0$, $r(\tau)$ has the upper boundary $r(\tau \to \infty) = r_\infty$, as already mentioned.

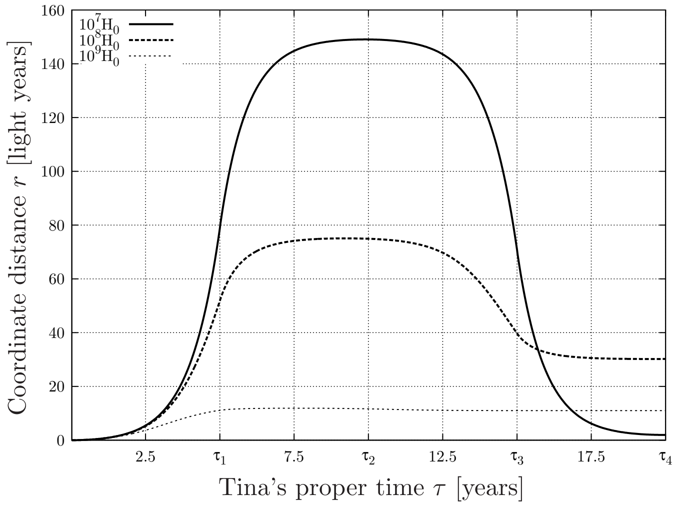

Figure 7. Coordinate distance r for trips with $\tau_1 = 5$ y, $\tau_3 = 15$ y in de Sitter space for very large Hubble constants.

4.2.2. Round trip with equal acceleration and deceleration times. In this section, we consider a simple trip, where Tina chooses her acceleration and deceleration times equally long, i.e. $\tau_3 = 3\tau_1$. As in flat space we choose $\tau_1 = 5$ y. To illustrate the influence of the expansion during such a short trip, we choose an extremely large Hubble constant. Specifically, we consider the cases $H = \{10^7 H_0, 10^8 H_0, 10^9 H_0\}$, where H_0 is the Hubble constant of our universe, cf (26).

Figure 7 shows Tina's radial coordinate $r(\tau)$ during these journeys. Even for a Hubble constant that is 10^7 times larger than that of our universe, the expansion has only minor influence on the course of Tina's journey, cf figure 2. In that case, she reaches her maximum radial coordinate distance $r \approx 149.06$ ly at proper time $\tau \approx 9.8898$ y. After $4\tau_1 = 20$ y, however, she has not returned yet, because $r \approx 1.9511$ ly. It is not surprising that $r(\tau_4) > 0$. As the spacetime expands, Tina cannot return to Earth on these journeys. Besides that, there are several other differences from Minkowski space. Figure 8 shows the elapsed coordinate time $t(\tau)$ during the same trips. The expansion rate has a strong influence on time dilation. As we have shown in (19), the maximum possible velocity β_∞ and the maximum value γ_∞ of t' become smaller when H increases. The same is also true for a round trip. Thus, the smaller the time dilation, the larger the expansion rate. In table 1, r_∞ and γ_∞ are compared for the expansion rates considered here. In figure 9, Tina's velocity $\beta = v/c$ during her trip is shown. Clearly, $\beta = 0$ already for some time $\tau < 2\tau_1$ and $\beta \neq 0$ at the end of the trip. In table 2, the elapsed coordinate times $t(\tau_4)$ after these journeys and the respective journey in Minkowski space, as well as the coordinate distance $r(\tau_4)$, the proper distance $l(\tau_4)$ and the velocity $\beta(\tau_4)$, are compared. For $H = 10^7 H_0$, the difference of coordinate time is small, but for even larger Hubble constants the elapsed time in de Sitter space is considerably smaller. On the other hand, Tina's velocity at the end of the trip is not zero and increases significantly with H, as does her proper distance. Her coordinate distance is also not zero with a more complex dependence on H. This is due to the fact that r_∞ becomes smaller for larger H and therefore so do the coordinate distances that Tina can reach during the respective journeys.

One further aspect can be seen in figure 8. From flat space, one would expect that $t(4\tau_1) = 2t(2\tau_1)$. Thus, the elapsed coordinate time after stage ④ is twice the time after stage

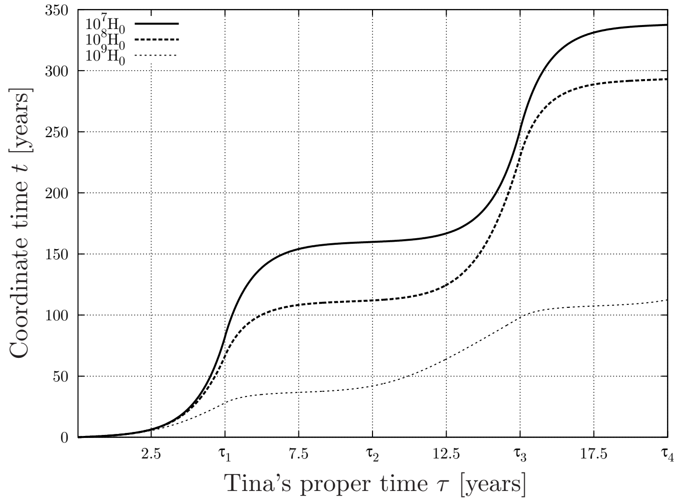

Figure 8. Coordinate time t for the same trips as in figure 7.

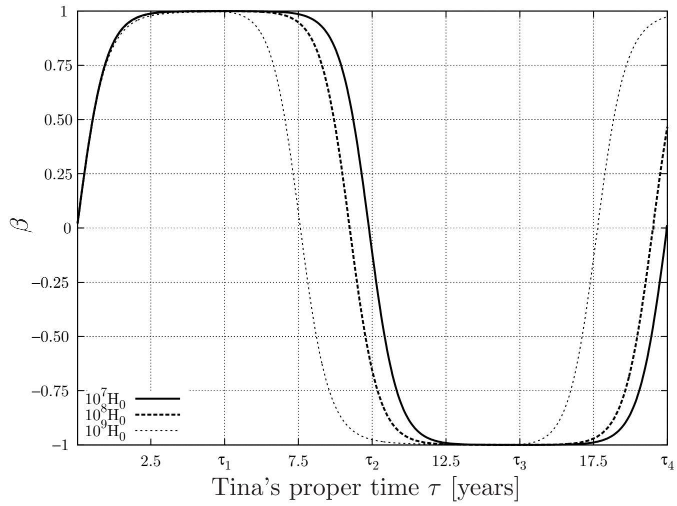

Figure 9. Velocity $\beta = v/c$ for the same trip as in figure 7. Contrary to the situation in flat space, we have $\beta \neq 0$ for $\tau = 20$ y.

Table 1. Maximum radial coordinate r_∞, Lorentz factor γ_∞ and velocity β_∞ for $\alpha = 9.81\ \mathrm{m\,s^{-2}} \approx 1.0326$ ly/ly^2 after infinite acceleration time in universes with different Hubble constants.

H	r_∞ (ly)	γ_∞	β_∞
H_0	1.3869×10^{10}	1.4322×10^{10}	$1 - 2.4376 \times 10^{-21}$
$10^7 H_0$	1.3860×10^{3}	1.4322×10^{3}	$1 - 2.4376 \times 10^{-7}$
$10^8 H_0$	1.3773×10^{2}	1.4322×10^{2}	$1 - 2.4375 \times 10^{-5}$
$10^9 H_0$	1.2995×10^{1}	1.4356×10^{1}	$1 - 2.4287 \times 10^{-3}$

Table 2. Elapsed time $t(\tau_4)$, coordinate distance $r(\tau_4)$, proper distance $l(\tau_4)$ and velocity $\beta(\tau_4)$ at the end of the trips in figure 7 and the equivalent trip in Minkowski space in figure 2.

H	$t(\tau_4)$ (y)	$r(\tau_4)$ (ly)	$l(\tau_4)$ (ly)	$\beta(\tau_4)$
Minkowski	338.36	0	0	0
$10^7 H_0$	337.55	1.95	2.49	0.0130
$10^8 H_0$	293.09	30.21	249.98	0.4668
$10^9 H_0$	112.45	10.99	36 456.47	0.9736

②, but here this is different. The reasons for these discrepancies are the different times Tina needs to accelerate to reach a certain velocity, and the time Tina needs to decelerate to come to rest again. We will evaluate this further in the following section.

4.2.3. Time to come to rest. In the preceding section, we have shown that a journey with four stages of the same duration is not an appropriate choice in de Sitter space. To perform a proper round trip, we must find the duration of stage ② necessary for a given stage ① such that, at its end, Tina is at rest again. Then, stages ③ and ④ have to be chosen in such a way that she returns to Earth and arrives there with zero velocity.

When Tina is at rest, we have $t' = 1$. With (14*b*) we can use this condition to calculate τ_{rest} and obtain

$$\tau_{\text{rest}} = \frac{1}{q} \ln \frac{S}{A_{\tau_1} D} \tag{27}$$

for the duration of stages ① + ②. It can easily be seen that

$$\tau_{\text{rest}} < 2\tau_1. \tag{28}$$

Thus, in de Sitter space, stage ② is always shorter than stage ①. In the limit $\tau_1 \to \infty$, we obtain

$$\lim_{\tau_1 \to \infty} (\tau_{\text{rest}} - \tau_1) = \frac{1}{q} \ln \frac{S}{H} \tag{29}$$

for the duration of stage ②. Thus, the time needed to come to rest after accelerating for arbitrarily long times has an upper boundary!

4.2.4. Maximum acceleration time. If Tina accelerates too long in the beginning of her trip, she can no longer return home to Earth afterwards. The condition for a possible return is $l(\tau_{\text{rest}}) < r_\infty$ for Tina's proper distance at the end of stage ②.

To make this point clear, we assume that Tina is on a journey and has covered a proper distance $l(\tau_{\text{rest}}) = r_\infty$. A transformation to new coordinates,

$$t \mapsto t - t(\tau_{\text{rest}}), \qquad r \mapsto [r - r(\tau_{\text{rest}})]\, \mathrm{e}^{Ht(\tau_{\text{rest}})}, \tag{30}$$

absorbs the previous expansion of the universe into the definition of our new coordinates. This transformation makes sense because of the form of the expansion factor $R(t) = \mathrm{e}^{Ht}$. Hence, we have $R(t_b)/R(t_a) = R(t_b - t_a)$ for arbitrary t_a and t_b, see also Tolman [22]. Thus, current proper distances in the old coordinates again equal coordinate distances in the new coordinates and the equations of motion are the same in the new coordinates. In these new coordinates, Tina's current position is $r = 0$ and Earth is located at $r_{\text{Earth}} = r_\infty$. Therefore, she can no longer return home. Hence, the set of all particles at rest at different radial coordinates r can be divided into four subsets at the beginning of the journey:

(i) particles beyond Tina's future light cone;
(ii) particles which Tina cannot reach because they have $r \geqslant r_\infty$;
(iii) particles which Tina can reach but where she cannot return to Earth afterwards, because when she arrives there, her proper distance to Earth is larger than r_∞;
(iv) particles which Tina can reach with $l < r_\infty$ and where she therefore can return to Earth afterwards.

For the times of departure $t_0 \neq 0$, the same classification can be made by replacing coordinate distances via $r \to r\,\mathrm{e}^{-Ht_0}$.

To find the maximum acceleration time $\tau_{1\max}$ that allows Tina to return home, we calculate the acceleration time for which Tina has exactly covered a proper distance

$$l(\tau_{\text{rest}}) = \mathrm{e}^{Ht(\tau_{\text{rest}})} r(\tau_{\text{rest}}) = r_\infty, \tag{31}$$

at the end of stage ②. Note that (31) is not a conditional equation for τ_{rest} but for $\tau_{1\max}$, with $\tau_{\text{rest}} = \tau_{\text{rest}}(\tau_{1\max})$! Solving this equation yields

$$\tau_{1\max} = \frac{1}{q} \ln\left(\frac{S}{H}\right). \tag{32}$$

Round trips are only possible for $\tau_1 < \tau_{1\max}$. For longer acceleration times, Tina cannot return to Earth. Using (32), we further obtain

$$\tau_{\text{restmax}} = \tau_{\text{rest}}(\tau_{1\max}) = \frac{1}{q} \ln\left(\frac{S}{D}\frac{Hq + \frac{\alpha^2}{c^2} - H^2}{2H^2}\right). \tag{33}$$

For $H = H_0$, we obtain $\tau_{1\max} = 22.6463$ y and $\tau_{\text{rest}}(\tau_{1\max}) = 44.6214$ y. For details on the calculation, see appendix D.

This situation can easily be illustrated using CE coordinates. To derive Tina's worldline $[\eta(\tau), \chi(\tau)]$ for infinitely long acceleration and on a round trip, we insert (20*a*), (21*a*) and (20), (21), respectively, into (4). Details on the calculation of τ_3 and τ_4 for a suitable round trip are given in the next section. Figure 10 shows Tina's worldline on a round trip (rt) with $\tau_1 = (1 - 10^{-8})\tau_{1\max}$, her worldline on a one-way trip (owt) where she accelerates away from Earth for all times and her future light cone at time $t = 0$ (lc) for a universe with $H = 5 \times 10^9 H_0$. In addition, the worldlines of particles at rest at Tina's EH $r_{\text{eh}} = c/H$ for $t = 0$, cf (8), at the α-horizon r_∞ and at $r_{\max} \equiv r(\tau_{\text{restmax}})$ are depicted. Here, the particle at $r_{\max}$ has the smallest distance that Tina cannot reach on a round trip. With (31) and (33), we arrive at

$$r_{\max} = r_\infty\, \mathrm{e}^{-Ht(\tau_{\text{restmax}})}. \tag{34}$$

For $H = 5 \times 10^9 H_0$ we obtain $r_{\max} = 0.40\,067 r_\infty$; for the general expression, see appendix E. Tina's future light cone intersects the worldline of the particle at the event horizon (eh) at $\eta = 0, t = \infty$; thus, this is the boundary of region 2 of points Tina can still send light signals to at time $t = 0$. Equally, the worldline for a trip with infinitely long acceleration (owt) intersects the worldline of the particle at r_∞ at $\eta = 0, t = \infty$, which borders region 3. On a round trip (rt) with τ_1 very close to $\tau_{1\max}$, Tina almost reaches the particle at $r_{\max}$, which is the boundary of region 4 of possible round trips.

4.2.5. A suitable round trip. In this section, we study how long Tina has to choose stages ③ and ④, to reach Earth again and come to rest there at the end of her journey.

The only difference between the outward trip and the return trip is the larger expansion factor $R[t(\tau_{\text{rest}})]$ instead of $R(0) = 1$. We use the same transformation as in the preceding section and consider Tina's proper distance $l(\tau_{\text{rest}})$ as a coordinate distance for $\tau = t = 0$.

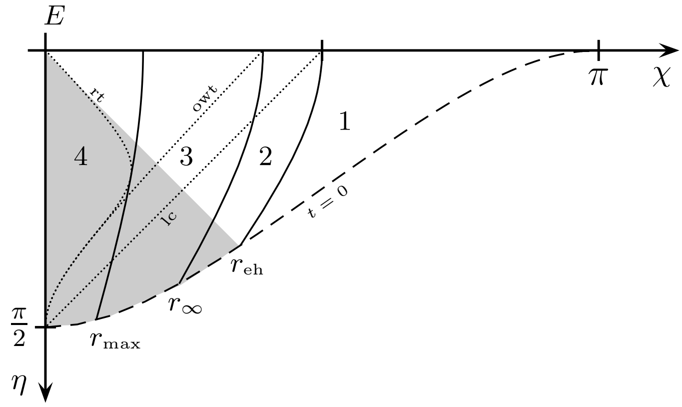

Figure 10. Situation in a de Sitter universe with $H = 5 \times 10^9 H_0$. The solid lines represent the worldlines of particles at r_{max}, r_∞ and r_{eh}. The dotted lines represent Tina's worldline on a round trip (rt) with $\tau_1 = (1 - 10^{-8})\tau_1$, a one-way trip (owt) where she accelerates away from Earth for all times, and her future light cone (lc) at time $t = 0$. All points with $r < r_{\mathrm{max}}$ can be reached by Tina on a round trip (region 4). On a one-way trip Tina can reach all points with $r < r_\infty$ (region 3) and she can send a light signal to all points with $r < r_{\mathrm{eh}}$ (region 2). Furthermore, a light signal emitted at $r = r_{\mathrm{eh}}$ at time $t = 0$ reaches Eric at $\eta = 0$, $t = \infty$. All points in region 1 are outside Tina's future light cone and Eric's backward light cone (grey-shaded region).

Then we can, in principle, calculate the proper duration $\tau_{1\mathrm{return}}$ for stage ③, which she needs to return home by calculating the time needed to cover the respective coordinate distance starting at $t = 0$. The equation

$$r[\tau_{\mathrm{rest}}(\tau_{1\mathrm{return}})] = d \tag{35}$$

with arbitrary $d < r_\infty$ is more difficult to solve than the similar equation (31) for l, where the factor e^{Ht} and the restriction on $d = r_\infty$ nicely simplify the resulting expressions. Therefore, this equation can only be treated numerically. The effects of expansion become especially clear for journeys with acceleration times close to $\tau_{1\mathrm{max}}$. As an example, we consider a universe with $H = 10^9 H_0$. From (32) we obtain $\tau_{1\mathrm{max}} = 2.6389$ y in this case. For our discussion, we consider three journeys, with acceleration times $\tau_1 = \{0.99\tau_{1\mathrm{max}}, 0.9999\tau_{1\mathrm{max}}, 0.99\,999\tau_{1\mathrm{max}}\}$ in stage ①. In table 3, these journeys are compared with respect to Tina's and Eric's elapsed times and Tina's maximum coordinate and proper distances at the end of stage ②. As the duration of stage ① is chosen very close to the maximum acceleration time, the time needed to return varies strongly with minimally different durations for stage ①. Figures 11–14 show $\beta(\tau)$, $t(\tau)$, $r(\tau)$ and $l(\tau)$ for these trips. Tina has to travel with very high velocity for a very long time on her way home, thus causing a large time dilation, see figure 11. Therefore, the elapsed coordinate time after the trip also increases strongly with minimal increase in initial acceleration time, see figure 12. The comparison of figures 13 and 14 shows the unimportance of the r-coordinate as a measure of distance. For most of the return trip, Tina's radial coordinate distance to Earth is very small. Her proper distance however has almost the same qualitative τ-dependence as the coordinate distance in flat space, cf figure 2.

4.2.6. A trip to the end of the universe. In flat Minkowski space, as already discussed by Müller *et al* [3], Tina could reach the most distant galaxies about 1.37×10^{10} ly away on a

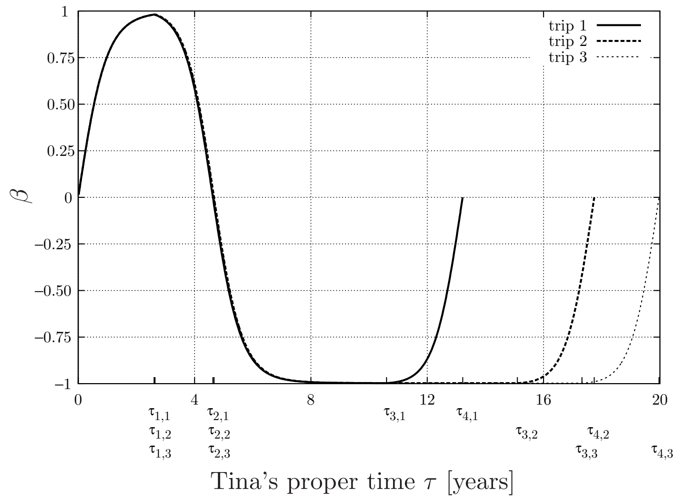

Figure 11. Velocity $\beta(\tau)$ for journeys with acceleration times $\tau_1 = 0.99\tau_{1\mathrm{max}}$ (trip 1), $0.9999\tau_{1\mathrm{max}}$ (trip 2) and $0.99\,999\tau_{1\mathrm{max}}$ (trip 3). For comparison, the respective times $\tau_{n,m}$ for the end of phase n in trip m are marked.

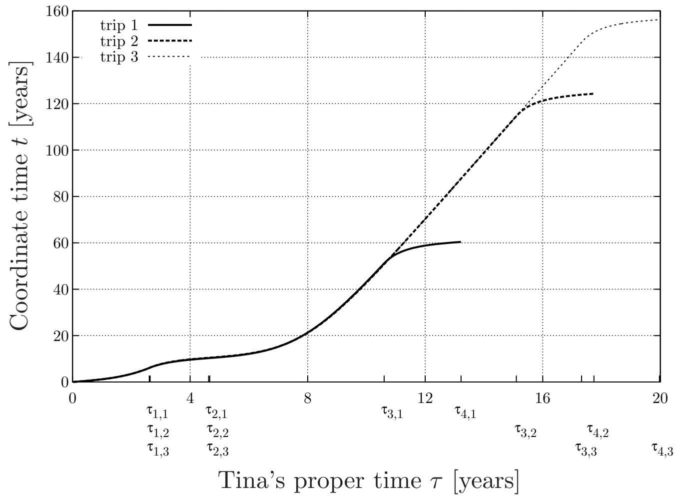

Figure 12. Coordinate time t for the same journeys as in figure 11.

trip with $\tau_1 \approx 22.635$ y. This can easily be reproduced by setting $\tau = 2\tau_1$ in (2*b*). When Tina has reached her destination, however, $t \approx 1.37 \times 10^{10}$ y have elapsed for Eric.

In a de Sitter universe, the expansion has an extremely large effect during such a long trip as is easily seen by evaluating

$$r_\infty = 1.38\,696 \times 10^{10}\,\text{ly}, \tag{36}$$

for $H = H_0$. Thus, the galaxies considered in flat space are out of reach for Tina; in fact they are beyond the EH, which differs only minimally from r_∞ in this case, as $\alpha/c \gg H_0$, cf the last paragraph in section 4.1. Therefore, we consider the quasar [23] 3C 324 with a distance

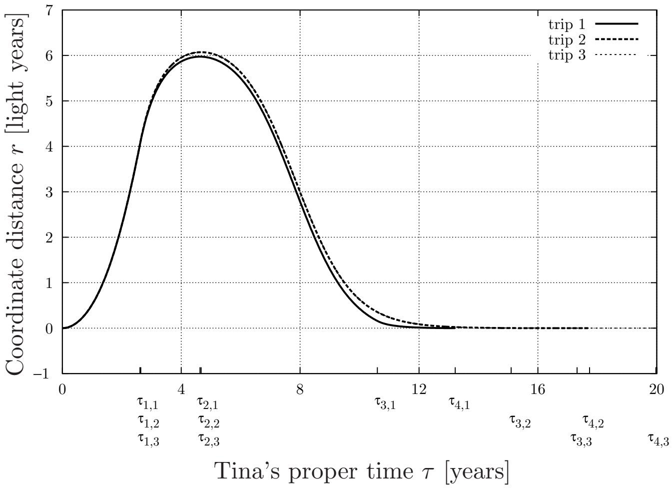

Figure 13. Coordinate distance r for the same journeys as in figure 11.

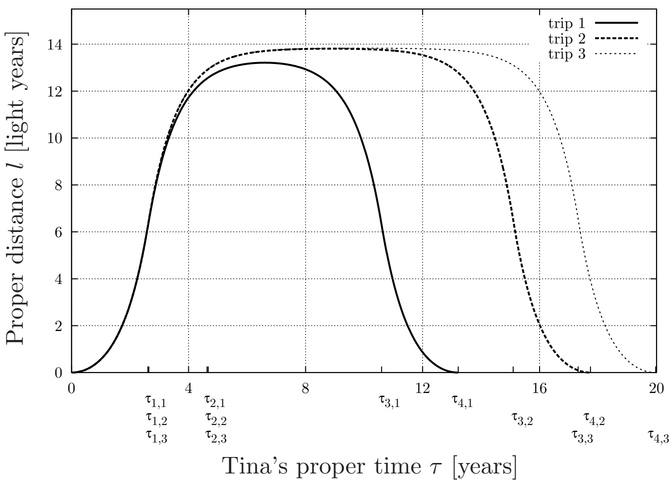

Figure 14. Proper distance l for the same journeys as in figure 11.

Table 3. Journeys with initial acceleration time τ_1 close to the maximum acceleration time $\tau_{1\text{max}}$ for $H = 10^9 H_0$. The table shows Tina's elapsed time τ_2 and Eric's elapsed time $t(\tau_2)$ for the outward journey, Tina's time τ_3 at the end of stage ③, Tina's time $\Delta\tau_{24} = (\tau_4 - \tau_2)$ and Eric's time $\Delta t_{24} = t(\tau_4) - t(\tau_2)$ for the return journey and for the round trip τ_4, $t(\tau_4)$, as well as the radial and proper distance $r(\tau_2)$, $l(\tau_2)$, which Tina covers during these journeys, compared to r_∞.

$\tau_1/\tau_{1\text{max}}$	τ_2 (y)	$\Delta\tau_{24}$ (y)	τ_3 (y)	τ_4 (y)	$t(\tau_2)$ (y)	Δt_{24} (y)	$t(\tau_4)$ (y)	$r(\tau_2)$ [r_∞]	$l(\tau_2)$ [r_∞]
0.99 000	4.6302	8.5862	10.60 391	13.2164	10.2857	50.1219	60.4076	0.46 173	0.96 930
0.99 990	4.6686	13.0742	15.10 422	17.7428	10.4866	113.8086	124.2952	0.46 935	0.99 969
0.99 999	4.6690	15.2990	17.32 912	19.9679	10.4885	145.7429	156.2314	0.46 942	0.99 997

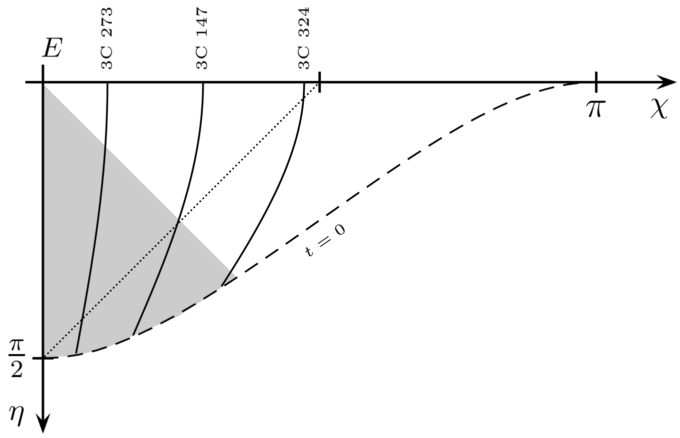

Figure 15. Worldlines of the quasars 3C 273, 3C 147 and 3C 324 in CE coordinates, $H = H_0$. The dotted line indicates Tina's forward light cone at $t = 0$. The grey-shaded region corresponds to Eric's backward light cone. The quasar 3C 324 is close to Tina's future light cone. Because of the very small Hubble constant, $r_\infty \lesssim r_{eh}$ and Tina can reach 3C 324, cf figure 10.

Table 4. Round trips in flat space. The table shows Tina's acceleration times τ_1 necessary to travel certain distances, her travel time τ_2 to the destination, the time τ_3 when she starts to decelerate on her way back to Earth and τ_4 for the round trip, as well as Eric's elapsed time $t(\tau_2)$ when she reaches her destination, and $t(\tau_4)$ when she is back at Earth.

Destination	Distance (ly)	τ_1 (y)	τ_2 (y)	τ_3 (y)	τ_4 (y)	$t(\tau_2)$ (y)	$t(\tau_4)$ (y)
Andromeda	2.56×10^6	14.32 048	28.64 097	42.96 145	57.28 193	$2.56\,000 \times 10^6$	$5.12\,000 \times 10^6$
3C 273	2.44×10^9	20.96 353	41.92 706	62.89 060	83.85 413	$2.44\,000 \times 10^9$	$4.88\,000 \times 10^9$
3C 147	6.44×10^9	21.90 340	43.80 681	65.71 021	87.61 362	$6.44\,000 \times 10^9$	$1.28\,800 \times 10^{10}$
3C 324	1.21×10^{10}	22.51 183	45.02 367	67.54 248	90.04 734	$1.20\,710 \times 10^{10}$	$2.41\,420 \times 10^{10}$

of approximately 1.21×10^{10} ly. (This is the distance a light ray emitted today would travel to the quasar if the expansion of the universe stopped today.) As less distant destinations we use the Andromeda Galaxy, which is approximately 2.56×10^6 ly away, see e.g. McConnachie [24], the quasar [25] 3C 273 with a distance of approximately 2.44×10^9 ly and the quasar [26] 3C 147, which is approximately 6.44×10^9 ly away. Table 4 shows how long Tina has to choose stage ① to reach these destinations in Minkowski space. See figure 15 for the quasers' worldlines in CE coordinates. To make the comparison with de Sitter space easier, we also list the times $\tau_2 = 2\tau_1$, $\tau_3 = 3\tau_1$ and $\tau_4 = 4\tau_1$ at the end of the respective stages. We also list how much time has elapsed for Eric when Tina has reached her destination, and when she has returned to Earth. Table 5 shows the same numbers in de Sitter space with $H = H_0$. Because of the expansion, stage ① is longer than in flat space. Stage ②, however, is almost equally long as in flat space as Tina can decelerate more quickly than she accelerates, see section 4.2.3. The same is also true for the return journey, to an even greater extent.

For a trip to the Andromeda galaxy and even to the quasar 3C 273, the deviation from the flat space journey is very small. At these destinations, Tina's proper distance from Earth is still well less than r_∞. For the quasar 3C 147, the deviations are very large, especially when comparing Eric's elapsed time in both spacetimes. This quasar is almost at the maximum

Table 5. The same numbers as in table 4 for round trips in de Sitter space with $H = H_0$. In addition, Tina's proper distance from Earth at her destination is listed.

Destination	τ_1 (y)	τ_2 (y)	τ_3 (y)	τ_4 (y)	$t(\tau_2)$ (y)	$t(\tau_4)$ (y)	$l(\tau_2)$ $[r_\infty]$
Andromeda	14.32 066	28.64 115	42.96 199	57.28 265	$2.56\,024 \times 10^6$	$5.12\,095 \times 10^6$	0.00 018
3C 273	21.15 091	42.11 445	63.49 791	84.64 882	$2.68\,367 \times 10^9$	$6.01\,429 \times 10^9$	0.21 348
3C 147	22.50 791	44.41 132	68.87 145	91.37 936	$8.65\,778 \times 10^9$	$3.66\,176 \times 10^{10}$	0.86 680
3C 324	24.48 999	47.00 182	–	–	$2.83\,312 \times 10^{10}$	–	6.71 124

distance that allows Tina to return to Earth; when she arrives there, her proper distance to Earth is $l \approx 0.86\,680 r_\infty$.

For 3C 324, only a one-way trip is possible; Tina cannot return to Earth afterwards. Also, Tina has to choose stage ① more than 2 years longer than in flat space. As she moves at the highest velocity during these 2 years, this rather small difference in acceleration time causes a huge difference on the trip. If Tina accelerates for the same time $\tau_1 = 24.48\,999$ y in Minkowski space, she covers a distance of $r = 9.30\,827 \times 10^{10}$ ly at the end of stage ②, which is almost eight times the distance with $\tau_1 = 22.51\,183$ y.

The huge influence of these additional 2 years can also be seen by comparing Eric's elapsed times in Minkowski space and de Sitter space. When Tina reaches 3C 324, more than twice the amount of time has elapsed in de Sitter space.

5. Communication between the twins

We imagine that Eric and Tina permanently send each other information about their respective current time τ or t. We study this situation both from Tina's and Eric's perspective. Concretely, we investigate when a signal, sent by Eric at time t_S, will be received by Tina, and when a signal that Eric receives at time t_R was sent by Tina. For Tina, we calculate the respective times τ_S and τ_R. Again, we compare our results with Minkowski space.

5.1. Infinitely long acceleration

First we consider a journey where Tina accelerates away from Earth for all times. In Minkowski space a light signal sent by Eric at time t_S is described by

$$r_L(t) = c(t - t_S). \tag{37}$$

This light signal reaches Tina when

$$r_L[t(\tau)] = r(\tau). \tag{38}$$

With the expressions in (1*a*) and (2*a*), we obtain

$$t_S = \frac{c}{\alpha}(1 - e^{-\frac{\alpha}{c}\tau_R}), \qquad \tau_R = -\frac{c}{\alpha}\ln\left(1 - \frac{\alpha}{c}t_S\right). \tag{39}$$

Obviously Tina can only receive messages that Eric sends at times $t_S < c/\alpha$.

Light signals sent by Tina at her proper time τ_S are described by

$$r_L(t) = r_{\text{flat}}(\tau_S) + ct_{\text{flat}}(\tau_S) - ct = \frac{c^2}{\alpha}(e^{\frac{\alpha}{c}\tau_S} - 1) - ct, \tag{40}$$

where we have used (1a) and (2a). In this case, we obtain

$$\tau_{\mathrm{S}} = \frac{c}{\alpha}\ln\left(1+\frac{\alpha}{c}t_{\mathrm{R}}\right), \qquad t_{\mathrm{R}} = \frac{c}{\alpha}\left(\mathrm{e}^{\frac{\alpha}{c}\tau_{\mathrm{S}}}-1\right). \tag{41}$$

Hence, Eric can receive every signal from Tina.

For radially moving light signals in de Sitter space ($\vartheta, \varphi = \mathrm{const}$), we have with (3)

$$\mathrm{d}r = \pm c\,\mathrm{e}^{-Ht}\,\mathrm{d}t. \tag{42}$$

Light signals sent by Eric are moving outwards ($\mathrm{d}r > 0$), and signals sent by Tina have $\mathrm{d}r < 0$. Thus, light signals sent by Eric at time t_{S} are described via

$$r_{\mathrm{L}}(t) = \frac{c}{H}\left(\mathrm{e}^{-Ht_{\mathrm{S}}} - \mathrm{e}^{-Ht}\right), \tag{43}$$

where the constant of integration is chosen so that $r(t_{\mathrm{S}}) = 0$. Tina receives these signals when

$$r_{\mathrm{L}}[t(\tau_{\mathrm{R}})] = r(\tau_{\mathrm{R}}). \tag{44}$$

Inserting (20a) into (43) yields

$$r_{\mathrm{L}}(\tau_{\mathrm{R}}) = \frac{c}{H}\left(\mathrm{e}^{-Ht_{\mathrm{S}}} - \frac{2q^2\,\mathrm{e}^{q\tau}}{\Psi(\tau_{\mathrm{R}})}\right). \tag{45}$$

Rearranging (44) then leads to

$$t_{\mathrm{S}} = -\frac{1}{H}\ln\left(2q\frac{\left(q-\frac{\alpha}{c}\right)\mathrm{e}^{q\tau_{\mathrm{R}}} + \frac{\alpha}{c}\frac{H}{S}}{\Psi(\tau_{\mathrm{R}})} + \frac{\alpha}{cS}\right) = \frac{1}{H}\ln\left(\frac{HS\,\mathrm{e}^{q\tau_{\mathrm{R}}} - \frac{\alpha}{c}\left(q-\frac{\alpha}{c}\right)}{\frac{\alpha}{c}H\,\mathrm{e}^{q\tau_{\mathrm{R}}} + S\left(q-\frac{\alpha}{c}\right)}\right), \tag{46a}$$

$$\tau_{\mathrm{R}} = \frac{1}{q}\ln\left(\frac{q-\frac{\alpha}{c}}{H}\,\frac{S+\frac{\alpha}{c}\,\mathrm{e}^{-Ht_{\mathrm{S}}}}{S\,\mathrm{e}^{-Ht_{\mathrm{S}}} - \frac{\alpha}{c}}\right), \tag{46b}$$

as a generalization of (39). With $\Psi(\tau)$ from (16a), we obtain

$$t_{\mathrm{Smax}} = \lim_{\tau\to\infty} t_{\mathrm{S}} = -\frac{1}{H}\ln\left(\frac{H}{c}r_{\infty}\right) = \frac{1}{H}\ln\left(\frac{c}{\alpha}S\right). \tag{47}$$

Only signals sent by Eric at times $t < t_{\mathrm{Smax}}$ can reach Tina. For $H \to 0$ we further obtain

$$\lim_{H\to 0} t_{\mathrm{Smax}} = \frac{c}{\alpha}, \tag{48}$$

in accordance with the result for flat space.

The dependence of t_{S} on H is very weak. For a universe with $H = H_0$, we find a deviation for t_{Smax} of around $1 : 10^{-21}$ from the flat space result; for $H = 10^9 H_0$, it is still only a difference of $1 : 8.1 \times 10^{-4}$. This is not surprising as we are on a very small timescale compared to times for which the expansion has a measurable effect.

In Minkowski space every signal sent by Tina eventually reaches Eric. In de Sitter space this is not true, since Tina eventually crosses Eric's EH. Tina's signals in de Sitter space are described via

$$r_{\mathrm{L}}(t) = \frac{c}{H}\left(\mathrm{e}^{-Ht} - \mathrm{e}^{-Ht(\tau_{\mathrm{S}})}\right) + r(\tau_{\mathrm{S}}), \tag{49}$$

instead of (40) and by expressions (20a) and (21a). We obtain

$$\tau_{\mathrm{S}} = \frac{1}{q}\ln\left(\frac{q+\frac{\alpha}{c}}{H}\,\frac{S-\frac{\alpha}{c}\,\mathrm{e}^{-Ht_{\mathrm{R}}}}{S\,\mathrm{e}^{-Ht_{\mathrm{R}}} + \frac{\alpha}{c}}\right), \tag{50a}$$

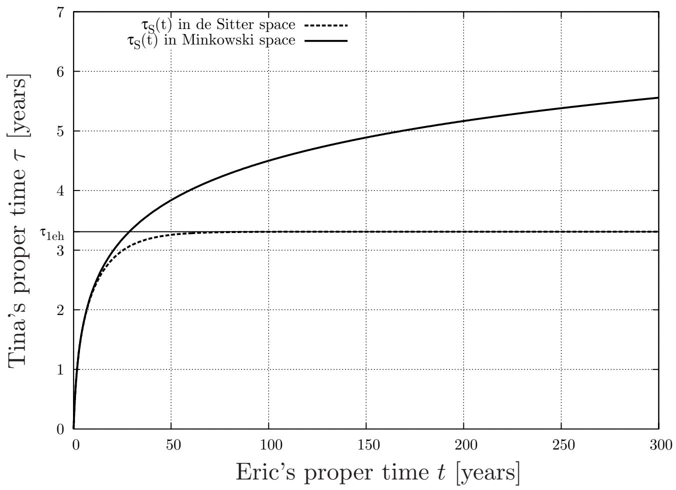

Figure 16. Eric's perspective: in Minkowski space, Eric has to wait very long for messages from Tina, but eventually he receives all signals from her. In de Sitter space ($H = 10^9 H_0$), Eric cannot receive signals that Tina has sent after τ_{1eh}. For all times, he therefore receives messages that Tina has sent at some earlier time.

$$t_R = -\frac{1}{H}\ln\left(2q\frac{\left(q+\frac{\alpha}{c}\right)e^{q\tau_S} - \frac{\alpha}{c}\frac{H}{S}}{\Psi(\tau_S)} - \frac{\alpha}{cS}\right) = \frac{1}{H}\ln\left(\frac{HS\,e^{q\tau_S} + \frac{\alpha}{c}\left(q+\frac{\alpha}{c}\right)}{S\left(q+\frac{\alpha}{c}\right) - \frac{\alpha}{c}H\,e^{q\tau_S}}\right), \qquad (50b)$$

instead of (41).

When Tina crosses Eric's EH, t_R diverges. Hence, we can use it to calculate the respective proper time by setting the argument of the logarithm equal to zero. The result of this calculation is

$$\tau_{1eh} = \frac{1}{q}\text{acosh}\left(1 + \frac{c}{\alpha}\frac{q^2}{H}\right) = \frac{1}{q}\ln\left[\frac{S}{H}\left(q\frac{c}{\alpha}+1\right)\right]. \qquad (51)$$

For $H = H_0$, Tina has to accelerate for 23.3176 y to cross Eric's EH; for $H = 10^9 H_0$, it takes her 3.3096 y.

In figures 16 and 17, we look at Eric's perspective in a Minkowski space and a de Sitter space with $H = 10^9 H_0$. For signals sent by Eric, the effect of the expansion is hard to recognize. For signals which he receives from Tina, the expansion has a large effect, when Tina approaches his EH and eventually moves behind it.

5.2. Communication during a round trip

To study communication during a round trip, the complete worldline of Tina has to be considered. The results are analogous to those in the preceding section. We present figures to illustrate this situation and omit the explicit mathematical expressions. For flat space, this problem has already been considered by Müller *et al* [3] specifically for a flight to Vega. Again we consider a trip with stage ① of $\tau_1 = 0.99\,999\tau_{max}$ in a universe with $H = 10^9 H_0$ and a trip in a flat universe with equally long stages ①–④ for comparison. This time we consider the situation from Tina's perspective. Figure 18 shows the situation in flat space, and figure 19 shows the situation in de Sitter space. In both cases, Tina receives very few signals at the beginning of her journey. This is easy to understand, as she accelerates away from Earth,

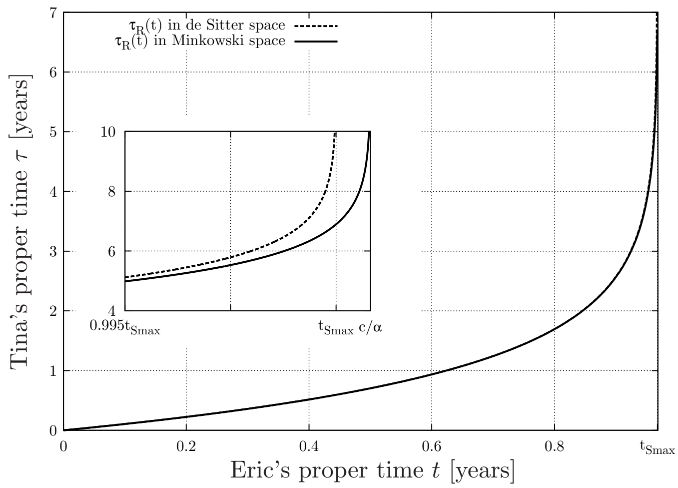

Figure 17. Eric's perspective: as Tina accelerates away from Eric, only messages that he sends before $t = c/\alpha$ in Minkowski space and before $t = t_{\mathrm{Smax}}$ from (47) in de Sitter space ($H = 10^9 H_0$) can reach Tina. To show the effect of the expansion, the region around t_{Smax} is also shown enlarged. Only signals sent at times $t \lesssim t_{\mathrm{Smax}}$ are received significantly later in de Sitter space. For $t \in [t_{\mathrm{Smax}}, c/\alpha)$, communication is only possible in Minkowski space.

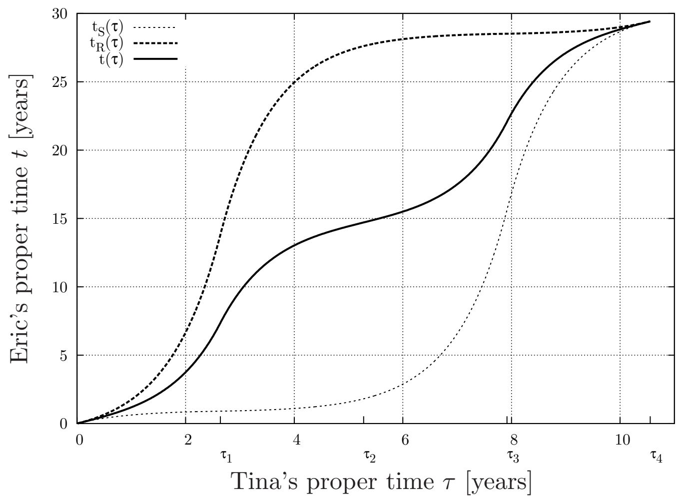

Figure 18. Tina's perspective in Minkowski space: at the proper time τ of Tina, the coordinate time is $t = t(\tau)$. A signal Tina receives at this time was emitted by Eric at time $t_S(\tau)$, and a signal she emits at that time will be received by Eric at the time $t_R(\tau)$.

the travel time of Eric's signals increases rapidly and time dilation additionally increases this effect. When she decelerates and starts to return to Earth, the rate of received signals increases. On the other hand, most of her signals reach Eric only shortly before she herself returns to Earth. The difference between the journeys in Minkowski space and de Sitter space is the large period in de Sitter space where the rate of received signals remains constant for Tina and also for Eric. Also, in flat space the travel time of the incoming light ray equals the travel time

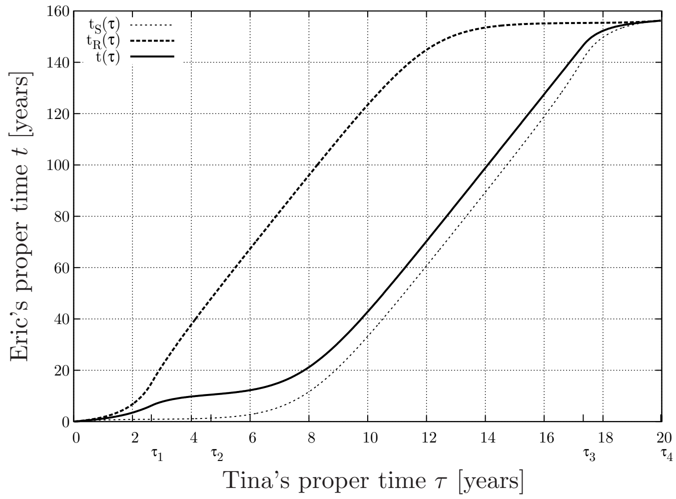

Figure 19. The same situation as in figure 18 but in de Sitter space with $H = 10^9 H_0$.

of the outgoing light ray, as they have to cover the same distance; thus, in figure 18 we always have

$$t_R(\tau) - t(\tau) = t(\tau) - t_S(\tau). \tag{52}$$

For $H = 10^9 H_0$ however, the deviations are very large, as Tina's signal has to travel a larger proper distance to reach Eric.

6. Summary

In this work, we have studied the extension of the twin paradox to de Sitter space. We showed that an expanding spacetime has a huge influence on long journeys, and we were able to quantitatively compare journeys in this spacetime with their counterparts in flat space, concentrating on the duration of the respective journeys, the possibility of communication during the journeys and the limitations that exist for round trips due to the expanding spacetime, which can make a return to the point of departure impossible.

Acknowledgment

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Appendix A. Hyperbolic motion in Minkowski space

In flat Minkowski space, the four-acceleration a^μ, see (12), for radial motion ($\vartheta, \varphi = \text{const}$) simplifies to

$$a^\mu = \frac{d^2 x^\mu}{d\tau^2}. \tag{A.1}$$

Therefore, (11) yields

$$\alpha^2 = -c^2 t''^2 + r''^2. \tag{A.2}$$

On the other hand, the constraint $g_{\mu\nu}u^\mu u^\nu = -c^2$ yields

$$r'^2 = c^2(t'^2 - 1). \tag{A.3}$$

Differentiating both sides logarithmically and multiplying with r' leads to

$$r'' = \frac{t't''}{t'^2 - 1} r'. \tag{A.4}$$

Combining (A.2), (A.3) and (A.4) yields the differential equation

$$t''^2 = \left(\frac{\alpha}{c}\right)^2 (t'^2 - 1), \tag{A.5}$$

with the solution $t'(\tau) = \cosh\left(\frac{\alpha}{c}\tau + c_0\right)$. Thus, with $t(0) = 0$ and $r(0) = 0$, we obtain

$$t(\tau) = \frac{c}{\alpha}\sinh\left(\frac{\alpha}{c}\tau\right), \qquad r(\tau) = \frac{c^2}{\alpha}\left[\cosh\left(\frac{\alpha}{c}\tau\right) - 1\right], \tag{A.6}$$

see (1*a*) and (2*a*). The derivation of (1*b*), (1*c*) and (2*b*), (2*c*) is straightforward.

Appendix B. Rindler's calculations

With the relevant Christoffel symbols [27]

$$\Gamma^r_{tr} = H, \qquad \Gamma^t_{rr} = \frac{H}{c^2}\,\mathrm{e}^{2Ht} \tag{B.1}$$

of the de Sitter space, Tina's four-acceleration a^μ for radial motion ($\vartheta, \varphi = \text{const}$) is given by

$$a^t = t'' + \frac{H\,\mathrm{e}^{2Ht}}{c^2} r'^2, \qquad a^r = r'' + 2Hr't', \tag{B.2}$$

and $a^\vartheta = a^\varphi = 0$. Inserting (B.2) into (11) and using the relations

$$r'^2 = c^2(t'^2 - 1)\mathrm{e}^{-2Ht}, \tag{B.3}$$

$$\frac{r''}{r'} = \frac{t't''}{t'^2 - 1} - Ht', \tag{B.4}$$

which follow from a similar calculation to that used to generate (A.3) and (A.4), Rindler derives the differential equation

$$\left(\frac{\alpha}{c}\right)^2 = \left\{t'^2(t'^2 - 1)\left(\frac{t''}{t'^2 - 1} + H\right)^2 - [t'' + H(t'^2 - 1)]^2\right\} \tag{B.5}$$

for t', cf (A.5). The substitution

$$t' = \cosh(z) \tag{B.6}$$

simplifies (B.5) to

$$\left(\frac{\alpha}{c}\right)^2 = [z' + H\sinh(z)]^2. \tag{B.7}$$

Integrating (B.7) and rearranging the terms yield

$$\frac{S + \frac{\alpha}{c}\tanh\left(\frac{z}{2}\right)}{D - \frac{\alpha}{c}\tanh\left(\frac{z}{2}\right)} = A\,\mathrm{e}^{q\tau}. \tag{B.8}$$

When Tina leaves the origin from rest at time $\tau_0 = 0$, we have $\beta(0) = 0$ and, therefore, $t'(0) = \gamma(0) = 1$; thus, $z_0 = 0$. Hence, $A_{\tau_0} = S/D$. Now t' can be calculated using the relation

$$\cosh(z) = \frac{1 + \tanh^2(z/2)}{1 - \tanh^2(z/2)} \tag{B.9}$$

and (B.8) as

$$t' = \frac{q(S\,\mathrm{e}^{2q\tau} + D)}{\Psi(\tau)}. \tag{B.10}$$

Integrating (B.10) and adjusting the constant of integration so that $t(0) = 0$ yields (20*a*). Using the positive root of (B.3) and the relations

$$\mathrm{e}^{-\mathrm{Ht}(\tau)} = \frac{2\mathrm{q}^2\,\mathrm{e}^{\mathrm{q}\tau}}{\Psi(\tau)}, \tag{B.11}$$

$$\sqrt{t'^2 - 1} = \frac{\alpha}{c}\frac{(\mathrm{e}^{q\tau} - 1)(S\,\mathrm{e}^{q\tau} + D)}{\Psi(\tau)} \tag{B.12}$$

together with the condition $r(\tau = 0) = 0$, Rindler arrives at (21*a*).

Appendix C. Extension for a decelerating observer

If Tina starts decelerating at $\tau = \tau_1$ by changing the proper acceleration via $\alpha \to -\alpha$, we must ensure that t' is continuous at $\tau = \tau_1$ and that r and t are differentiable. The condition of continuity for t' is fulfilled if

$$z_1 = \mathrm{acosh}[t'(\tau_1)], \tag{C.1}$$

cf (B.6). Using the trigonometric relations

$$\tanh(z/2) = \frac{\mathrm{e}^z - 1}{\mathrm{e}^z + 1}, \qquad \mathrm{acosh}(x) = \ln(x + \sqrt{x^2 - 1}), \tag{C.2}$$

we obtain

$$\tanh(z_1/2) = \frac{t'(\tau_1) + \sqrt{t'(\tau_1)^2 - 1} - 1}{t'(\tau_1) + \sqrt{t'(\tau_1)^2 - 1} + 1} = 1 - 2\frac{\Psi(\tau_1)}{\Omega(\tau_1)}, \tag{C.3}$$

with $\Psi(\tau_1)$ and $\Omega(\tau_1)$ given in (16*a*) and (16*d*), respectively. Inserting (C.3) into (B.8) yields the constant of integration A_{τ_1}, cf (16*b*), and furthermore t' for $\tau > \tau_1$. The calculation for $\tau = \tau_3$ is performed in the same way and yields B_{τ_3}, cf (16*c*). Combining our result with Rindler's expressions for $\tau \leqslant \tau_1$, we obtain the piecewise definition of t' in (14). Integrating (14*a*) and (14*b*) and choosing the additional constants of integration to ensure continuity of $t(\tau)$, we obtain (20). To calculate $r(\tau)$ we also use the positive root of (B.3) and the expressions analogous to (B.12), choose the emerging constant of integration properly and derive (21).

Appendix D. Maximum acceleration time

In order to determine the maximum acceleration time $\tau_{1\max}$ via

$$l(\tau_{\mathrm{rest}}) = r_\infty, \tag{D.1}$$

we evaluate $r(\tau_{\mathrm{rest}})$ and $\mathrm{e}^{Ht(\tau_{\mathrm{rest}})}$ with τ_{rest} given in (27), and using (20*b*) and (21*b*), where

$$\tau_{\mathrm{rest}} = \tau_{\mathrm{rest}}(\tau_{1\max}), \tag{D.2}$$

and obtain

$$e^{Ht(\tau_{\text{rest}})} = A_{\tau_{1\max}} \frac{D}{S} \frac{\Psi(\tau_{1\max})}{\Psi_A(\tau_{1\max})}, \tag{D.3}$$

$$r(\tau_{\text{rest}}) = \alpha \frac{\Psi_A(\tau_{1\max})}{\Psi(\tau_{1\max})} \frac{1}{HDA_{\tau_{1\max}}} + \mathcal{K}(\tau_{1\max}) + r_\infty. \tag{D.4}$$

Multiplying these expressions, subtracting r_∞ from both sides of (D.1) and taking out a factor $\Psi_A(\tau_{1\max})^{-1}$ yields

$$A_{\tau_{1\max}} D\Psi(\tau_{1\max})\,[\mathcal{K}(\tau_{1\max}) + r_\infty] = 0. \tag{D.5}$$

With $A_{\tau_1}, \Psi(\tau_1) \neq 0$, we have

$$\mathcal{K}(\tau_{1\max}) + r_\infty = 0 \tag{D.6}$$

as the defining equation for $\tau_{1\max}$.

To prove that $\tau_{1\max}$ given in (27) is a solution of this equation, we insert (32) into (D.6). With (22*b*) and the intermediate results

$$\Psi(\tau_{1\max}) = \frac{q^2}{H}(3q + 5H), \tag{D.7}$$

$$A_{\tau_{1\max}} = 2\frac{H^2}{Hq + \frac{\alpha^2}{c^2} - H^2}, \tag{D.8}$$

this can easily be shown.

Appendix E. Maximum coordinate distance during a round trip

As discussed in section 4.2.4, a round trip is only possible for destinations with

$$r < r_{\max} = r_\infty\, e^{-Ht(\tau_{\text{restmax}})}. \tag{E.1}$$

To calculate $e^{-Ht(\tau_{\text{restmax}})}$, we use the results in (D.7) and (D.8) and additionally

$$\Psi_A(\tau_{\text{rest}}) = 2q^2, \tag{E.2}$$

which is true for any τ_{rest} from (31) and

$$\Psi(\tau_{\text{restmax}}) = \frac{A^2_{\tau_{1\max}} D^2 S}{H} + \frac{2}{H} D^2 S A_{\tau_{1\max}} - HD. \tag{E.3}$$

With (D.7), (D.8), (E.3) and (20*b*), we obtain

$$e^{-Ht(\tau_{\text{restmax}})} = \frac{S}{2} \frac{2H^4 - H^2\frac{\alpha^2}{c^2} - 2Hq^3 - 3\frac{\alpha^4}{c^4}}{(3q + 5H)(H^2 - \frac{\alpha^2}{c^2} - Hq)q^2}, \tag{E.4}$$

and hence $r_{\max}$.

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