Touching ghosts: observing free fall from an infalling frame of reference into a Schwarzschild black hole

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Abstract

The problem of communication between observers in the vicinity of a black hole in a Schwarzschild metric is considered. The classic example of an infalling observer Alice and a static distant mother station (MS) is extended to include a second infalling observer Bob, who follows Alice in falling towards the event horizon. Kruskal coordinates are introduced to analyse this situation, and the pedagogic value of introducing this alternative coordinate system is demonstrated by their ability to provide a graphically based solution to and illustration of this problem. The counterintuitive result that is obtained, which will be of interest to those studying general relativity at an introductory level at final year undergraduate or at graduate level, is that Bob appears to reach a ghostly image of Alice at the event horizon.

1. Introduction

One of the most appealing features of the gravitational field yielded by a spherically symmetric, static mass M is the fact that it is an exactly solvable problem both in classical and in relativistic mechanics. The meaning of exact solvability is that the gravitational field itself is known (in relativistic mechanics, the metric tensor) and the equation of motion of a test particle can be solved. That is why it is often used as an example of a relativistic equation of motion in the framework of Schwarzschild coordinates. The simplest problem in this case is radial free fall. Despite the exact solvability and simplicity of the problem (see e.g. [1]) one can still find some new, unexpected and even intriguing features. One such unexpected characteristic (unknown to the best of our knowledge) occurs when discussing the problem of communication with

an object falling towards the event horizon in this case. A well-known feature of radial fall towards the event horizon is that the *coordinate time* (the time that would be measured by a static distant observer) taken to reach the event horizon by a freely falling object is infinite (see e.g. [2]). This feature is accompanied by unlimited redshift: the frequency ratio of electromagnetic signals coming from the falling object and recorded by a distant observer at rest tends to zero as the infalling observer approaches the event horizon. In principle, such signals will be recorded for an infinite future duration; in practice, they will disappear from the screens of rest observers due to the sensitivity limits of recording devices. On the other hand, an infalling observer reaches the event horizon in a finite proper time (the time measured by the infalling observer). This may be regarded as the most dramatic manifestation of time-dilation-like phenomena [2].

Bearing in mind such a dramatic difference between the measurements of an observer at rest and an infalling observer, one may consider an intermediate situation, and ask: what would be the observations made by two infalling observers, A (Alice) being followed by B (Bob)? Alice crosses the event horizon in 'the infinite future' in terms of coordinate time, but in a finite proper time, as measured by her—what would be Bob's perception of this event? How much later (behind Alice) will Bob himself cross the event horizon, according to his proper time? Would this be shortly after Alice appears to? We will discuss these questions here because the key answer turns out to be rather surprising: Bob, while continuously in receipt of signals from Alice, finds her crossing the horizon at the instant when he himself crosses the horizon. This counterintuitive result is found to be much less surprising when studied in more detail. An especially illuminating description of this effect may be given using so-called Kruskal coordinates [3, 4], and it has pedagogic value to introduce the use of these specialized co-ordinates to a wider audience by addressing this particular problem of interest. By focusing primarily on the qualitative character of this observation rather than on its quantitative aspects, one can obtain an appealing and simple description of this nontrivial observation. Although Schwarzschild black holes have been known for over 90 years [5, 6]—but only widely accepted for some 40 years—and more exotic variants such as charged [6] and/or rotating black holes [7, 8] have been described, including coverage in introductory texts [9, 10] as well as more rigorous textbooks [11, 12], further pedagogical value of the approach taken here is given by the fact that despite not being a new scientifically original result, the current treatment provides a new perspective on a well-known feature of these objects, namely free fall in Schwarzschild spacetime. As a matter of fact, this new perspective might be added to the list of phenomena unknown in the 'classical world' that are accessible to us only by means of relativistic physics.

2. Radial fall in Schwarzschild spacetime

Let us start by defining the framework for our considerations. One may consider a static, spherically symmetric mass M of radius R. The properties of the gravitational field outside of this gravitating body, $r > R$, are described in terms of the metric tensor. The metric tensor expressed in Schwarzschild coordinates turns out to be diagonal and the line element is then expressed as

$$d\tau^2 = \left(1 - \frac{r_S}{r}\right) c^2 dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\varphi^2$$
$$\equiv g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2, \tag{1}$$

where the Schwarzschild radius is denoted $r_S = \frac{2GM}{c^2}$. Solution (1) holds above the *event horizon*, $r > r_S$.

The motion of a test particle within a Schwarzschild spacetime (1) may be described in a rigorous manner by reference to conservation laws. There are two conserved quantities: energy and angular momentum. The planar motion (angular momentum conservation) may be considered to occur in an equatorial plane, $\theta = \frac{\pi}{2}$, and hence the equation of motion simplifies in this case to the 4-velocity condition

$$g_{00}(u^0)^2 + g_{11}(u^1)^2 + g_{33}(u^3)^2 = 1, \tag{2}$$

where $u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$ denotes the α-component of the velocity vector.

Hereafter, only radial fall implied by $u^3 = 0$ is considered. Then energy conservation (a result of the time independence of the system, see [1, 13]) gives an explicit dependence for the time-component velocity vector

$$u_0 = g_{00}u^0 = \varepsilon. \tag{3a}$$

This in turn (see equation (2)) leads to the obvious solution for the radial velocity component:

$$u^1 = \mp\sqrt{\varepsilon^2 - g_{00}}. \tag{3b}$$

The parameter $\varepsilon = \sqrt{g_{00}(r_0)}$ can be shown to be related to the energy and is determined by the initial conditions: r_0 denotes the starting point of the radial fall, assuming free fall beginning from rest. One can also consider massless test particles such as photons. In this case, the corresponding tangential 4-vector $k^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}$ is (by definition) a null vector:

$$g_{\alpha\beta}k^\alpha k^\beta = 0, \tag{4}$$

where σ is an auxiliary parameter of a photon world line. Components of the wave vector for radial fall, found by use of the energy conservation condition (cf equations (3a)) and (4), take the form (see the discussion in [14])

$$k^0 = \frac{\omega_\infty}{g_{00}} \qquad k^1 = \mp\omega_\infty. \tag{5}$$

The $\mp$ sign corresponds to incoming and outgoing massless test particles (light rays).

By using equations (3a), one can calculate the time elapsed to reach the event horizon. The proper time, i.e. the time as measured by the infalling observer, is defined by equation (3b),

$$\mathrm{d}\tau = -\frac{\mathrm{d}r}{\sqrt{g_{00}(r_0) - g_{00}(r)}} = -\frac{\mathrm{d}r}{\sqrt{\frac{r_S}{r} - \frac{r_S}{r_0}}}, \tag{6a}$$

while the coordinate time is defined by equations (3a) and (3b):

$$\mathrm{d}t = -\frac{\sqrt{g_{00}(r_0)}}{\left(1 - \frac{r_S}{r}\right)\sqrt{g_{00}(r_0) - g_{00}(r)}}\,\mathrm{d}r = -\frac{\sqrt{1 - \frac{r_S}{r_0}}}{\left(1 - \frac{r_S}{r}\right)\sqrt{\frac{r_S}{r} - \frac{r_S}{r_0}}}\,\mathrm{d}r. \tag{6b}$$

As direct calculation shows [2, 15] an infalling object reaches the event horizon

(a) after a finite amount of proper time (according to the infalling observer) and
(b) after an infinite time (according to a distant, static observer).

Indeed, by using the substitution $x = \sqrt{\frac{r_0}{r} - 1}$, one can integrate equation (6a)

$$\tau - \tau_0 = -\int_{r_0}^r \frac{\mathrm{d}\bar{r}}{\sqrt{\frac{r_S}{\bar{r}} - \frac{r_S}{r_0}}} = r_0\sqrt{\frac{r_0}{r_S}}\left(\arctan x + \frac{x}{1 + x^2}\right) \tag{7a}$$

to find that the infalling observer starting from r_0 reaches the event horizon in a finite (proper) time,

$$\Delta\tau = \sqrt{\frac{r_0}{r_S}}(r_0 \arctan x_S + r_S x_S) < \sqrt{\frac{r_0}{r_S}}\left(r_0 \frac{\pi}{2} + r_S x_S\right),$$

$$x_S = \sqrt{\frac{r_0}{r_S} - 1}.$$

On the other hand, the coordinate time period, found from equation (6*b*),

$$t - t_0 = -\int_{r_0}^{r} \frac{d\bar{r}}{\left(1 - \frac{r_S}{\bar{r}}\right)\sqrt{\frac{r_S}{\bar{r}} - \frac{r_S}{r_0}}} = -r_S[\ln(x_S - x) - \ln(x_S + x)]$$

$$+ 2r_S x_S \arctan x + r_S x_S \left(\arctan x + \frac{x}{1+x^2}\right) \tag{7b}$$

tends to infinity as r approaches the event horizon,

$$\Delta t \propto -r_S \ln(x_S - x) \xrightarrow{\; x \to x_s \;} \infty.$$

This is a time-dilation-like effect and it is a well-known feature of a Schwarzschild black hole. Less well known is the frequency ratio relation accompanying mutual signal exchange between Alice and her 'mother station', MS, located at r_0. Namely, one finds that the frequency ratio is redshifted in both cases. MS receives (r) signals emitted (e) by Alice, and the frequency ratio is defined as a scalar product ratio [16]:

$$\frac{\omega_{MS}^r}{\omega_A^e} = \frac{(k^\alpha u_\alpha)_{MS}}{(k^\alpha u_\alpha)_A} = \frac{(g_{00}k^0 u^0)_{MS}}{(g_{00}k^0 u^0 + g_{11}k^1 u^1)_A} = \frac{\frac{\omega_\infty}{\sqrt{g_{00}(r_0)}}}{\omega_\infty u^0 \left(1 + \frac{g_{11} u^1}{u^0}\right)_A}. \tag{8}$$

This may be expressed in terms of Alice's local velocity, v_A (as measured by a local observer at rest [17]), via

$$u_A^0 = \frac{1}{\sqrt{g_{00}(r_A)}} \frac{1}{\sqrt{\left(1 - \frac{v_A^2}{c^2}\right)}}. \tag{9}$$

Then it takes a simple form, namely

$$\frac{\omega_{MS}^r}{\omega_A^e} = 1 - \frac{v_A}{c} \tag{10a}$$

and is critically redshifted, i.e. it tends to zero [18], because v_A tends to c as the event horizon is approached. (There is a simple argument that v_A tends to c independently of the starting point r_0: an immediate consequence of the energy conservation (3*a*) and equation (9) is that

$$\frac{\sqrt{g_{00}(r_A)}}{\sqrt{\left(1 - \frac{v_A^2}{c^2}\right)}} = \text{const}$$

is conserved, hence v_A tends to c when $r_A \to r_S$ [17].) In turn, Alice receives signals issued by MS, *universally* redshifted,

$$\frac{\omega_A^r}{\omega_{MS}^e} = \frac{1}{1 + \frac{v_A}{c}}. \tag{10b}$$

This ratio tends to 1/2 when Alice approaches the event horizon, independently of the starting position. This asymmetry in communication eventually becomes singular: MS receives Alice's signals emitted only above the event horizon, whereas Alice will receive signals from

MS continuously above and below the event horizon. In this sense two-way communication is disrupted. One may ask how two-way communication would be disrupted between two falling observers, Alice and Bob, with Bob following Alice. Both Alice and Bob would reach the horizon within a finite proper time period so their two-way communication must be broken within a finite proper time.

To study this more easily one needs a change of reference frame. Namely, the considerations above have been performed within Schwarzschild coordinates, which produce a singularity at the event horizon, $r = r_S$. A different frame of reference may be deployed, devoid of this defect.

3. Kruskal coordinates

Schwarzschild coordinates (1) yield a singularity at the event horizon: $g_{00}(r_S) = 0$. This singularity is termed a coordinate singularity, and it does not appear in other, carefully chosen systems of coordinates. There are various coordinate systems free of this singularity [1, 2]. One of them is the Kruskal (or Kruskal–Szekeres) coordinate frame [2–4]. Coordinates in this frame are usually denoted as v, u, θ, φ and the relation $(t, r) \to (v, u)$ is the following:

$$
v = \left| \frac{r}{2M} - 1 \right|^{1/2} e^{\frac{r}{4M}}
\begin{cases}
\sinh\left(\dfrac{t}{4M}\right) & r > r_S \\[2mm]
\cosh\left(\dfrac{t}{4M}\right) & r > r_S
\end{cases}
$$

$$
u = \left| \frac{r}{2M} - 1 \right|^{1/2} e^{\frac{r}{4M}}
\begin{cases}
\cosh\left(\dfrac{t}{4M}\right) & r > r_S \\[2mm]
\sinh\left(\dfrac{t}{4M}\right) & r > r_S.
\end{cases}
\tag{11}
$$

The last two coordinates are those of the Schwarzschild angular coordinates.

The transformation (11), above the event horizon $r > r_S$, is described by the following matrix:

$$
\frac{\partial (v, u)}{\partial (t, r)} =
\begin{bmatrix}
\dfrac{1}{4M} u & \dfrac{1}{4M g_{tt}} v \\[3mm]
\dfrac{1}{4M} v & \dfrac{1}{4M g_{tt}} u
\end{bmatrix}.
\tag{12a}
$$

Inverting this matrix results in the inverse transformation:

$$
\frac{\partial (t, r)}{\partial (v, u)} = D^{-1}
\begin{bmatrix}
\dfrac{1}{4M g_{tt}} u & -\dfrac{1}{4M g_{tt}} v \\[3mm]
-\dfrac{1}{4M} v & \dfrac{1}{4M} u
\end{bmatrix},
\tag{12b}
$$

where

$$
D = \det \left| \frac{\partial (v, u)}{\partial (t, r)} \right| = \frac{1}{16M^2 g_{tt}} (u^2 - v^2) = \frac{r}{32M^3} e^{\frac{r}{2M}}.
$$

By performing transformation (12b), one can find the corresponding metric tensor in Kruskal coordinates

$$
K_{\gamma \delta} = g_{\alpha \beta} \frac{\partial x^\alpha}{\partial y^\gamma} \frac{\partial x^\beta}{\partial y^\delta},
\tag{13}
$$
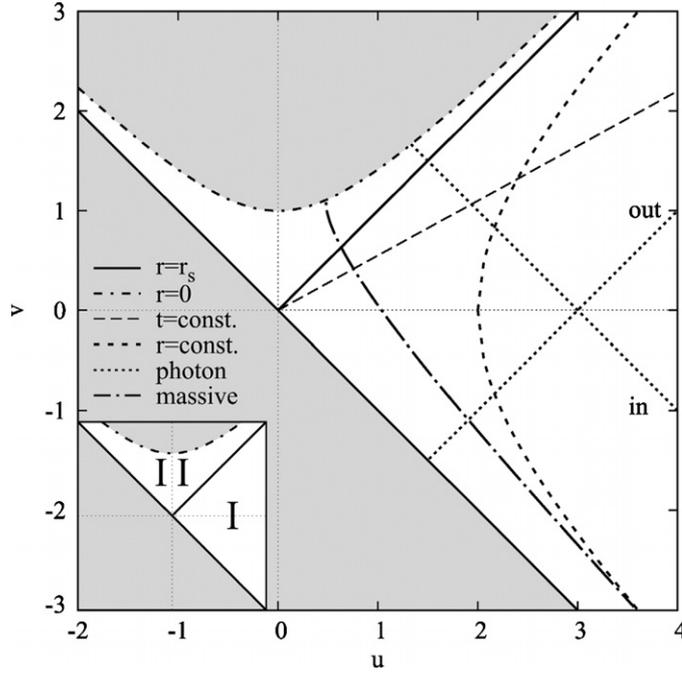
Figure 1. Kruskal coordinates v–u and radial geodesics. Manifold $u + v > 0$, covers black hole exterior (I) and interior (II). Hyperbolas with asymptotes $u = \pm v$ represent 'surfaces' of constant radius r; straight lines, passing through the origin, represent 'surfaces' of constant time t. Ingoing and outgoing radial null geodesics are $\mp45°$ straight lines. A geodesic for a massive particle is sketched schematically.

that does not reveal singular behaviour. In fact, the line element, expressed in Kruskal coordinates, is

$$\mathrm{d}\tau^2 = K\,(\mathrm{d}v^2 - \mathrm{d}u^2) - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\varphi^2, \tag{14}$$

where $K = \frac{32M^3}{r}\,\mathrm{e}^{-\frac{r}{2M}}$.

The most appealing features of Kruskal coordinates are: the nonsingular behaviour of the metric tensor and the fact that radial geodesics, ingoing and outgoing, are represented in the v–u plane as perpendicular, straight lines at $-45°$ and $+45°$, respectively (see below). The meaning of v–u coordinates becomes clearer when considering specific hypersurfaces— lines in this case (see figure 1). Namely, world lines of fixed distance, $r = $ const, are hyperbolas, $u^2 - v^2 = C$; space-like curves, $t = $ const are straight lines, originating at $v = u = 0$. Therefore, the region (I) bounded in figure 1 by the two straight lines: $v = -u$ corresponding to $r = r_S$ and $t = -\infty$ and $v = u$ corresponding to $r = r_S$ and $t = \infty$ represents the Schwarzschild coordinate system (1).

In the range $r > r_S$, applying transformation (12b) to solutions (3) and (5), one obtains the equations for geodesics. Geodesics are arranged in the v–u plane, and velocity vector components are then given as follows:

$$U^v = \frac{r}{4M(r - 2M)}\left(u\varepsilon - v\sqrt{\varepsilon^2 - \frac{(r - 2M)}{r}}\right) \tag{15a}$$
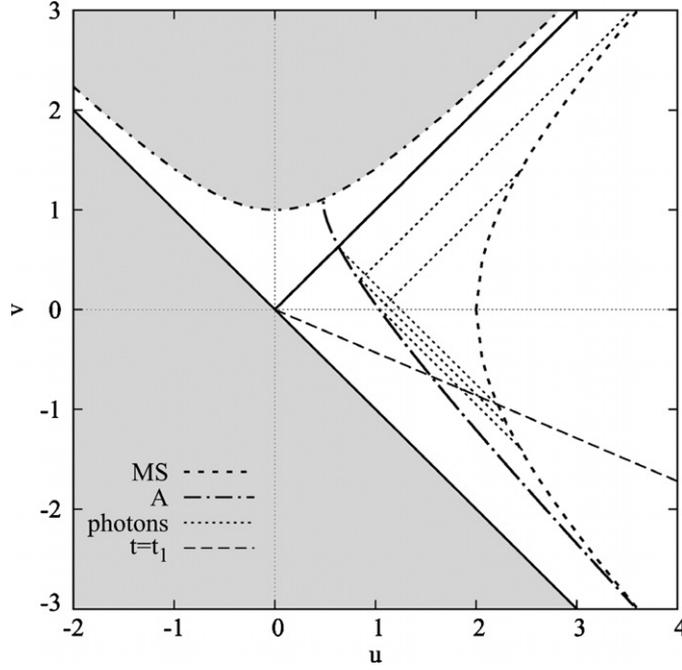
Figure 2. Exchange of radial signals between Alice and MS. Alice records (in principle) the signals from MS above and below the horizon; MS records only those signals sent above horizon.

$$U^u = \frac{r}{4M(r - 2M)}\left(v\varepsilon - u\sqrt{\varepsilon^2 - \frac{(r - 2M)}{r}}\right), \tag{15b}$$

where ε is a constant of motion (see equation (3a)). Let us emphasize that the velocity components (15) are well defined in the whole range $0 < r < \infty$. The wave vector components are

$$k_v = \frac{4M\Omega}{u^2 - v^2}(u \mp v) \tag{16a}$$

$$k_u = -\frac{4M\Omega}{u^2 - v^2}(v \mp u). \tag{16b}$$

These null geodesics are straight lines, at $\mp 45°$ to the x-axis, where the label $\mp$ stands for ingoing (outgoing) rays (see figure 1).

Applying these tools to the information exchange between Alice and the MS (see figure 2), one can reproduce qualitatively the well-known result: Alice receives messages from MS throughout her journey towards the event horizon and also below the event horizon. MS will receive messages sent by Alice from above the event horizon, throughout all the infinite future. Due to formal technical limitations, these messages, when sufficiently redshifted, will cease to arrive as their intensity will not be sufficiently high.

What may be easily deduced from figure 2 is that MS can, in principle, infer the last instant at which a message sent to Alice could produce a response that can be received asymptotically by MS in the future, namely the instant $t = t_1$. However, signals from Alice suffer unlimited redshift and as claimed above, at some stage they cease to be recorded by MS.
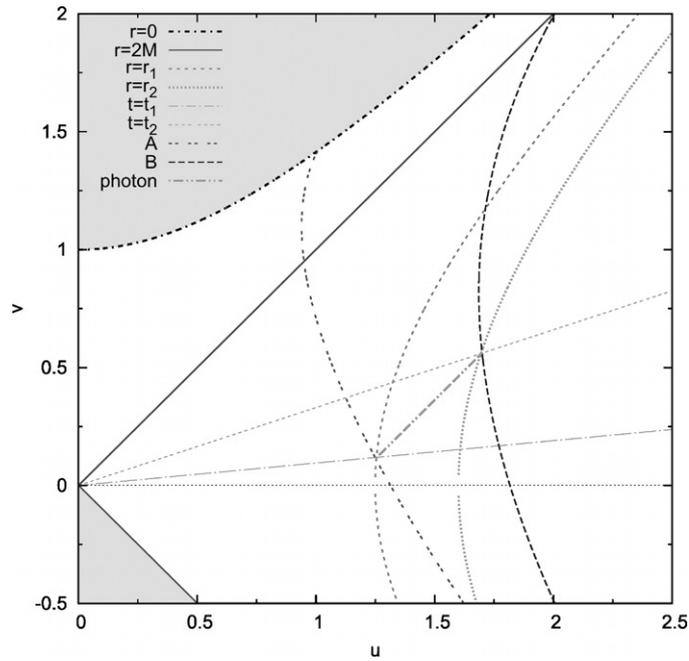
Figure 3. Bob receives a signal at $t = t_2$, $r = r_2$ from Alice crossing $r = r_1$ at $t = t_1$.

4. How does Bob 'see' Alice crossing the event horizon?

One may ask about communication between Alice and Bob, both falling towards the event horizon. As static observers, MS (and all other observers arranged radially at rest), receives messages from Alice continuously throughout the infinite future; the question is, how will the observer Bob, who is also going to cross the event horizon in a finite proper time τ_B, 'see' Alice approaching and crossing the event horizon? Instead of giving a direct answer, let us first consider a trivial question: how does Bob see Alice reaching and crossing a specific point at coordinate $r = r_1$? If Alice and Bob are communicating by means of electromagnetic signals then, Alice, when crossing $r = r_1$, at some $t = t_1$ sends a signal (back) to Bob. In Kruskal coordinates this signal is a straight line, $+45°$ that hits Bob's world line at some later instant $t = t_2$, at $r = r_2$ (see figure 3). One can use proper time notation: Alice's and Bob's world lines may be parameterized with their own proper times τ^A and τ^B, respectively, and consequently respective proper times representing the crossing instant $t = t_1$ (emission instant) and receiving instant $t = t_2$ would be fixed as τ_1^A and τ_2^B, respectively.

The situation of crossing the event horizon is, however, a special case. Approaching the event horizon, Alice emits signals continuously: those signals reach Bob continuously. But as can be seen in figure 4 those signals are recorded by Bob, who is also approaching the event horizon, until the very *instant of crossing* the event horizon himself! It looks like the information that Alice crosses the event horizon reaches Bob only at the event horizon.

This is obviously a counterintuitive outcome: it seems that Bob, approaching the event horizon can see Alice all the time, and moreover, that he is going to collide with her.
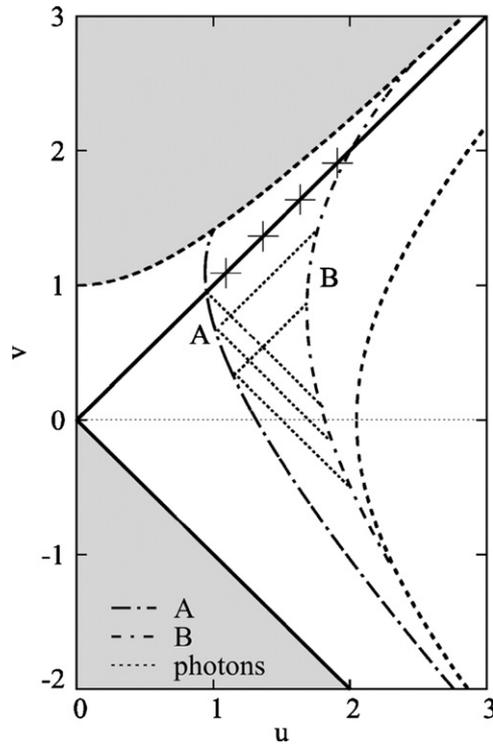
Figure 4. Bob receives signals from Alice until he, himself reaches the event horizon. This is an asymptotic process as no signal can be emitted at the horizon (there can be no 'trapped' signal).

5. Discussion and remarks

Geodesic radial fall in Schwarzschild spacetime does not seem to yield any controversies, at least above the event horizon. Nevertheless, despite being an exactly solvable problem, it still reveals interesting behaviour—namely, the dramatic discrepancy between a rest observer's and an infalling observer's perception of the time taken to reach the event horizon. A manifestation of this discrepancy is the disruption of communication: an infalling observer continues to receive electromagnetic signals from a rest observer positioned above the event horizon, even when they drop below the event horizon (i.e. within the interior of a black hole). On the other hand, the rest observer can only receive signals from the infalling observer while the latter remains above the event horizon. As may be deduced from the above, this does not lead to an information paradox: the infalling observer, from the perspective of the rest observer, disappears into the latter's infinite future. It should be pointed out though, that Alice appears to take forever to reach r_S. So intuitively one may expect that light rays, sent by MS at any future coordinate time, will reach her before she crosses the horizon. In fact, this is not the case. Only signals sent up to $t = t_1$ will do so, and responses to these may be received in the limit of the limit of infinite coordinate time. And that is a direct consequence of the aforementioned divergence of time perception.

However, as discussed in this paper, it is both interesting and instructive to explore another perspective—what will be perceived by a second infalling observer, Bob, who follows a first infalling one, Alice? Bob reaches the event horizon in a finite time, so for him the argument

of the infinite future does not apply. The analysis of observations made by Bob has been described here in a very qualitative manner, suitable for pedagogic purposes, by means of Kruskal coordinates. Paradoxically, those coordinates, devised mainly to match the interior of the black hole with its exterior, have been used here solely above the event horizon in order to illuminate the issue. The issue itself turns out to be the following: Bob continues to receive signals until the very last instant before crossing the event horizon. It would appear to him that he is going to collide with Alice. More precisely, it would appear that Bob's recording location approaches Alice's emission location, eventually, overlapping on the event horizon itself. Hence the allusion in the title of this paper to touching ghosts.

It should be noted that this conclusion is, in a sense, inevitable. Bob, in chasing Alice, has to receive signals until he himself crosses the event horizon. This may be proven by considering a counterexample. Suppose that Bob noted that Alice had already crossed the event horizon—this means that he is no longer receiving signals from her. Then he could emit a signal back to observers at rest letting them know that Alice had crossed the event horizon, and this signal would be received at a finite coordinate time. But this would then lead to a contradiction, since Alice's crossing of the event horizon cannot be recorded at a finite coordinate time as noted in the discussion above.

Although it would have been possible to draw this conclusion directly as noted above, it is more natural and possesses greater pedagogic value to do it in the deductive manner that we have followed in this paper.

What happens at the event horizon and what happens below the event horizon? These questions are beyond the scope of the current discussion, but one can provide here two impressions resulting from studies undertaken elsewhere. First, it seems that one cannot emit an outgoing signal at the event horizon; such a signal *would be* trapped at the horizon. Second, it appears that in the case of Schwarzschild spacetime, the black hole interior represents a region where there are two possible communication scenarios. In one of these scenarios, signals emitted by Alice 'towards' Bob—'outgoing' signal—are recorded by Bob below their emission location. This is in a sense an extension of the above discussion. The other scenario of communication in the black hole interior seems to be even more exotic.

The fascinating consequences of this for interaction between real objects provide grounds for further discussion.

References

[1] Hartle J B 2003 *Gravity: An Introduction to Einstein's General Relativity* (Reading, MA: Addison-Wesley) chapter 9
[2] Misner Ch W, Thorne K S and Wheeler J A 1972 *Gravitation* (San Francisco: Freeman) chapter 25
[3] Kruskal M D 1960 *Phys. Rev.* **119** 1743
Szekeres G 1960 *Publ. Mat. Debrecen* **7** 285
[4] Grib A A and Pavlov Yu V 2009 Is it possible to see the infinite future of the Universe when falling into a black hole? arXiv:0906.1442 [gr-qc]
[5] Schwarzschild K 1916 *Sitz. Preuss. Akad. Wiss. Berlin* **189**
[6] Reissner H 1916 *Ann. der Phys.* **355** 106
Nordström G 1918 *Kon. Nederland. Akad. Wet. Proc.* **20** 1238
[7] Kerr R P 1963 *Phys. Rev. Lett.* **20** 237
[8] Newman E T *et al* 1965 *J. Math. Phys.* **6** 918
[9] Frolov V P and Novikov I D 1998 *Black Hole Physics: Basic Concepts and New Developments* (Dordrecht: Kluwer)
[10] Hawking S 2001 *The Universe in a Nutshell* (New York: Bantam Books)
[11] Chandrasekhar S 1983 *The Mathematical Theory of Black Holes* (Oxford: Oxford University Press)
[12] Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Space-Time* (Cambridge: Cambridge University Press)
[13] Weinberg S 1972 *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (New York: Wiley)

[14] Okun L B, Selivanov K G and Telegdi V L 2000 *Am. J. Phys.* **68** 115
[15] Lindquist R W and Wheeler J A 1957 *Rev. Mod. Phys.* **29** 432
[16] Narlikar J V 1994 *Am. J. Phys.* **62** 903
[17] Radosz A, Augousti A T and Ostasiewicz K 2009 *Phys. Lett.* A **373** 801
[18] Augousti A T, Radosz A and Ostasiewicz K 2011 *Eur. J. Phys.* **32** 535