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Dissipation, voltage profile and Lévy dragon in a special ladder network

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Abstract

A ladder network constructed by an elementary two-terminal network consisting of a parallel resistor–inductor block in series with a parallel resistor–capacitor block sometimes is said to have a non-dispersive dissipative response. This special ladder network is created iteratively by replacing the elementary two-terminal network in place of the resistors. In this paper, it will be demonstrated that, in fact, non-dispersive dissipative response conclusion of this special ladder network is not accurate for steady-state condition. Furthermore, the voltage profile of this special ladder network exhibits a fractal form called the Lévy C curve or Lévy dragon in a certain condition. Therefore, this special ladder network may be called the Lévy ladder network. This ladder network might be interesting for physics and electrical engineering students and they may encounter when dealing with this network series and parallel combinations of impedances.

1. Introduction

Ladder networks are often encountered in college education when dealing with series and parallel combinations of impedances or admittances, such as in filter design and in wave propagation on transmission lines [1]. These networks can also be related to continued fractions in the theory of numbers, which is one of the oldest and the largest branches of pure mathematics. Furthermore, continued fractions may have fractal nature. Therefore, study of ladder networks has been the interest of many researchers from diverse disciplines. Physics and electrical engineering students may encounter ladder networks in their curriculum and hence, they as well as instructors may benefit by understanding these networks correctly.

Ladder networks are created by cascading elementary network blocks. An infinite number of these blocks constitutes an infinite ladder network. Input impedance of such an infinite ladder network can be calculated either by considering that the impedance does not change when a bock is added to the ladder or by starting from the terminating end, cascading the blocks and finally taking the block number to infinity. Both procedures will give the same input impedance if the latter converges [2, 3]. The latter assumes a steady state ac solution for the input impedance. With this assumption, the input impedance of the network may not converge under certain conditions. Therefore, one needs to be cautious when interpreting these two solutions [4].

In this paper, a special ladder network constructed by an elementary two-terminal network consisting of a parallel resistor–inductor block in series with a parallel resistor–capacitor block will be investigated in terms of power dissipations and voltage profiles of the resistors at the boundary of the network. This special ladder network is sometimes treated as having non-dispersive dissipative response [5]. Reference [5] numerically iterated this special network using recursive relation and concluded that regardless of the frequency and initializations, starting with a small resistance at this boundary, the network generates a finite resistance. The author calls the situation an anomaly because when infinitely iterated, it gives an essentially reactive network and yet provides dissipation. Dissipation of the network will be tested without using the recursive relation in this paper. Basically, the nodal equations for the ladder network will be developed and voltages across the resistors will be used to find the total power dissipation in the resistors. Furthermore, the voltage profile across the resistors leads to a fractal form and the characteristic of the fractal form will also be discussed in the paper.

The rest of the paper is organized as follows. The following section presents the recursive relation of the ladder network, investigates the fixed points and explains the stability of these fixed points. The third section develops the node equations of the ladder network to calculate the voltages across the resistors. The fourth section studies the dissipation and the voltage profile of the resistors for different size ladder networks and for different values of resistors. This section also probes into the similarity between the voltage profile of the ladder network and the fractal called the Lévy C curve or Lévy dragon. Conclusions are given in the last section.

2. Recursive relation and fixed points of the ladder network

Consider an elementary two-terminal network consisting of a parallel resistor–inductor block in series with a parallel resistor–capacitor block as shown in figure 1(a). The terminal impedance of the network in figure 1(a) will be equal to $\sqrt{L/C}$ provided that the values of resistors in the network are $R = \sqrt{L/C}$, independent of the driving source frequency. A ladder network can be constructed by replacing the elementary two-terminal network in place of the resistors iteratively. The network is shown in figure 1(b) after one replacement.

As an alternative to two-terminal representation, the network may be considered as a three-port network having one input and two output ports as shown in figure 2. The output ports are connected to the input ports of two three-port networks iteratively to create a ladder network. Current–voltage relations of the ports can be written in matrix form and recursive relations can be described by the multiplication of these matrices. Using port relations in matrix form may complicate the problem unnecessarily. Therefore, the two-terminal representation of the network illustrated in figure 1(a) will be used in this study.

A recursive relation can be written easily by using the two-terminal representation to express the impedance of the ladder network. In figure 3, Z_{n+1} impedance of the two-terminal network is expressed in terms of Z_n as

$$Z_{n+1} = f(Z_n) = \frac{\mathrm{i}\omega L Z_n}{\mathrm{i}\omega L + Z_n} + \frac{Z_n}{1 + \mathrm{i}\omega C Z_n}. \tag{1}$$

Initial impedance is specified as $Z_0 = R$ in the recursive relation. By examining the recursive relation given by (1), it can be easily verified that the input impedance of the ladder network

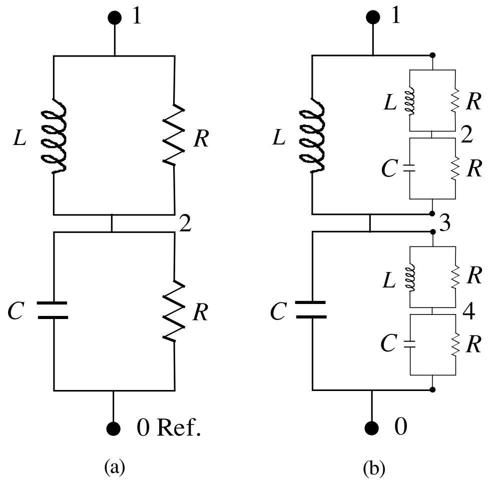


Figure 1. The elementary two-terminal network (a) and the resulting network after one iteration (b).

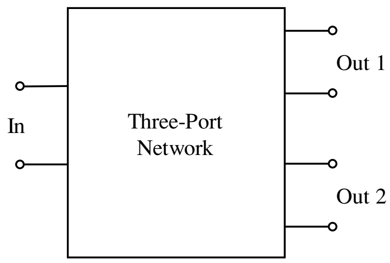


Figure 2. The three-port network representation.

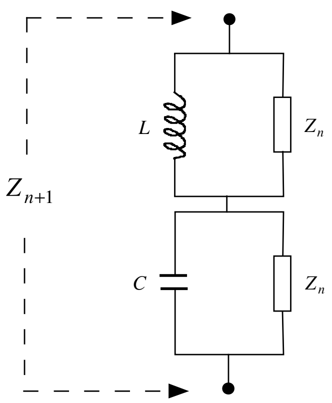


Figure 3. Network to describe the recursive relation between Z_{n+1} and Z_n.

will not converge if the initial impedance is purely imaginary or if its value is infinite (open circuit). For $Z_0 = \infty$ and $\omega = 1/\sqrt{LC}$, the network is in the resonant condition for the driving frequency and the input impedance is equal to zero.

Recursion described by (1) has three fixed points: $Z_1^* = 0$, $Z_2^* = \sqrt{L/C}$ and $Z_3^* = -\sqrt{L/C}$. Note that the fixed points are independent of the driving frequency. The stability of the three fixed points can be examined by evaluating the derivative of the function $f(Z)$ at the fixed points. The derivative is given by

$$f'(Z) = \frac{\omega^2 L^2}{(\omega L - \mathrm{i}Z)^2} - \frac{1}{(\omega C Z - \mathrm{i})^2}. \tag{2}$$

The fixed point is stable for $|f'(Z)| < 1$, unstable for $|f'(Z)| > 1$ and neutral for $|f'(Z)| = 0$. The first fixed point $Z_1^* = 0$ is unstable since $|f'(0)| = 2$. The second and the third fixed points are stable for nonzero finite values of L, C and ω as stated by

$$|f'(\pm\sqrt{L/C})| = \left| \frac{LC\omega^2 - 1}{(\sqrt{LC}\omega \mp \mathrm{i})^2} \right| < 1. \tag{3}$$

The stable fixed point Z_3^* is negative valued and therefore, does not correspond to a passive network.

In [5], equation (1) is numerically iterated for different initializations to show that the network impedance converges to $\sqrt{L/C}$ regardless of the frequency and initializations, and concluded that starting with an arbitrarily small resistance at the boundary, the network generates a finite resistance expressed by $\sqrt{L/C}$. Reference [5] describes this situation as an anomaly because a dissipative response is achieved from a non-dispersive network component. It points out the similarity of Nyquist–Johnson noise power generated by the two shunt resistors combined to give a noise output at the terminal equal to that for a single resistance. Another example is given in fluid turbulence. Energy fed at the large-scale eddy is cascaded away progressively to smaller-scale eddies to ultimately dissipate at the distant smallest scale whose dissipation rate is independent of the viscosity.

The recursive relation described by (1) does not reveal the dissipation across the resistors in the network. Therefore, dissipated power from the resistors for steady state will be calculated by using the nodal equations in the following section. Total power loss in the resistors will be compared to the injected power of the source to test the conclusion reached by [5].

3. Steady-state nodal equations for the ladder network

The analysis of the ladder network in sinusoidal steady state (ac analysis) will be carried out using the node-admittance matrix approach [6]. We will consider that the network is driven by a sinusoidal current source connected at node 1. The ladder network after n recursive iterations is shown in figure 4. This network has a total of $m+1$ nodes including the reference node which is labelled as zero. The other nodes are labelled from 1 to m as shown in figure 4. Let us call the voltages of these nodes with respect to the reference node node voltages. Using Kirchhoff's laws, root-mean-square (rms) values of the node voltage vector $\mathbf{V} = [V_1, V_2, \ldots, V_m]^T$ can be related to rms values of the node current source vector $I = [I_1, I_2, \ldots, I_m]^T$ through the node-admittance matrix $\mathbf{Y}$ as

$$\mathbf{I} = \mathbf{YV}. \tag{4}$$

For ac analysis, the entries of (4) are complex numbers in general. The node current source vector has only one entry in the first row since the only current source I is connected to node 1 in the network as shown in figure 4. Node voltages can be found by solving m simultaneous linear equations with complex coefficients expressed by

$$\mathbf{V} = \mathbf{Y}^{-1}\mathbf{I} = \mathbf{ZI}, \tag{5}$$

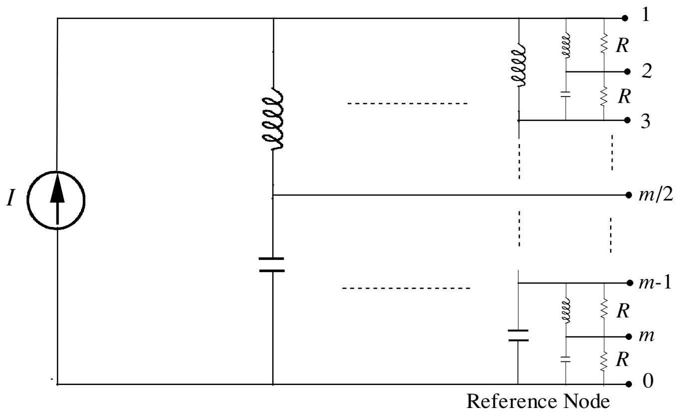


Figure 4. Ladder network after n iterations.

where $\mathbf{Z}$ is the node impedance matrix. The first column of the node impedance matrix is multiplied by the injected current at node 1 to determine the node voltages, that is, the node voltages become $V_i = Z_{i,1}I$. The voltages across resistors can be calculated by the difference of node voltages given by $V_i - V_{i+1}$. The total power dissipation P of the resistors at the boundary of the ladder network yields

$$P = \sum_{k=1}^{m} \frac{|V_k - V_{k+1}|^2}{R} = |I|^2 \sum_{k=1}^{m} \frac{|Z_{k,1} - Z_{k+1,1}|^2}{R}, \tag{6}$$

where $V_{m+1} = V_0 = 0$, which is the voltage of the reference node, and $|I|$ is the magnitude of current I. For infinitely large ladder network, the summation on the right side of (6) must converge to $\sqrt{L/C}$ for $R > 0$ since this is the positive fixed point of the network, that is,

$$\lim_{m\to\infty} \sum_{k=1}^{m} \frac{|Z_{k,1} - Z_{k+1,1}|^2}{R} = \sqrt{\frac{L}{C}}. \tag{7}$$

An analytical approach to show that equation (7) is correct may be difficult. Therefore, a numerical investigation based on the node-admittance matrix will be used to reveal the voltage profile of the resistors and to find the power dissipation of the network.

The node-admittance matrix can be constructed directly from the network. The $Y_{k,k}$ admittance in the principal diagonal of $\mathbf{Y}$ is found by adding all admittances connected to node k in the network. Off-diagonal admittance $Y_{k,j}$ is found by taking the negative value of the connected total admittances between nodes k and j. The node-admittance matrix $\mathbf{Y}$ is symmetrical around the principal diagonal and it is a square matrix of the size $m \times m$. One can develop an algorithm straightforwardly to form the node-admittance matrix from the procedure described here for the ladder network in figure 4.

The node-admittance matrix of the network shown in figure 1(b) can be constructed as an example. Let us consider that the admittance of the inductors is $Y_L = 1/\mathrm{i}\omega L$ and the admittance of the capacitors is $Y_C = \mathrm{i}\omega C$ in figure 1(b). The number of nodes in the network is equal to four excluding the reference node. Therefore, the size of the node-admittance matrix equals 4×4 and the matrix is given by

$$\mathbf{Y} = \begin{bmatrix} 2Y_L + 1/R & -Y_L - 1/R & -Y_L & 0 \\ -Y_L - 1/R & Y_L + Y_C + 2/R & -Y_C - 1/R & 0 \\ -Y_L & -Y_C - 1/R & 2Y_L + 2Y_C + 2/R & -Y_L - 1/R \\ 0 & 0 & -Y_L - 1/R & Y_L + Y_C + 2/R \end{bmatrix}. \tag{8}$$

The number of nodes in the network will grow geometrically. For example, there are three nodes for the elementary network given in figure 1(a) and five nodes for figure 1(b). In the tenth iteration of the recursive relation of (1), the number of nodes in the network will increase to 1025. Therefore, the size of the node-admittance matrix will get very large with a small iteration number.

4. Dissipation, voltage profile and Lévy dragon

Numerical investigation of voltages and dissipated powers across the resistors for different ladder network sizes can be easily done using the node-admittance approach described in the previous section. In the node-admittance approach, the size of the matrix will double, whereas the memory requirement will be quadrupled when the ladder network size is increased by one. Therefore, the maximum size of the ladder network that can be numerically investigated will be limited by the memory of the computer used. In this study, the maximum size of the ladder network investigated is 10 and therefore, the size of the node-admittance matrix is 1024 × 1024, that is, $m = 2^n$.

A program is written to construct the node-admittance matrix automatically. Once the node-admittance matrix is constructed, the inverse is taken and the complex valued node voltages are found. Certainly, node voltages will be a function of injected current to node 1 and for simplicity, the injected current is chosen to be 1 A. Therefore, the voltage at node 1 will converge to $\sqrt{L/C}$ for sufficiently large ladder network sizes. This value is the fixed point of the ladder network and if the value of the resistors is chosen to be that value, for every size of the network, voltage at node 1 will be equal to $\sqrt{L/C}$.

In figure 5, for the fixed point of $R = \sqrt{L/C}$, voltages across the resistors are drawn in the complex plane to reveal the voltage profile of different size ladder networks for $L = 1$ H, $C = 1$ F, $R = 1\ \Omega$, $\omega = 1$ rad s^{-1}. Figure 5(a) illustrates the voltage profile across the two resistors for the ladder network of size one, $n = 1$, shown in figure 1(a). There are two resistors of 1 Ω and the voltage across each resistor is drawn in a complex plane as shown in figure 5(a). The resistor connected between node 2 and the reference node has a voltage given by 0.5 − i0.5 V and the resistor connected between node 1 and node 2 has a voltage of 0.5 + i0.5 V. In each resistor, power dissipation is 0.5 W. Thus, the total power dissipation of the two resistors is equal to 1 W. The voltage profile of the resistors when the size of the network is increased to 2 ($n = 2$) is shown in figure 5(b). Each of the four resistors dissipates 0.25 W. Figures 5(c) and (d) show the voltage profiles of the ladder networks of sizes 4 and 8, respectively. For the numerical values used in this example, voltage magnitude across each resistor is identical, causing a symmetrical picture in the vertical direction. For identical voltage magnitudes, the power dissipation of each resistor in the network can be calculated by $1/2^n$. The voltage magnitudes across the resistors will be different in general and the voltage profiles can be experimented with different frequencies of driving current source of 1 A. For example, when the frequency is increased, the voltage magnitudes will decrease across the resistors in figure 4 when progressed from node 1 to the reference node because inductors dominate the upper part of the ladder network. Conversely, the voltage magnitudes will increase from node 1 to the reference node when the frequency is reduced. In both cases, symmetrical structure of the voltage profile no longer exists. However, the total power dissipation in the ladder network will not change.

The effect of the values of the resistors on the voltage profile in the network is also studied numerically for $n = 10$. The values used for the study are the same as the previous example, that is, $L = 1$ H, $C = 1$ F, $\omega = 1$ rad s^{-1}. The values of the resistors connected at the boundary are changed between 0.01 Ω and 10 Ω to observe voltage profiles. Figure 6(a)

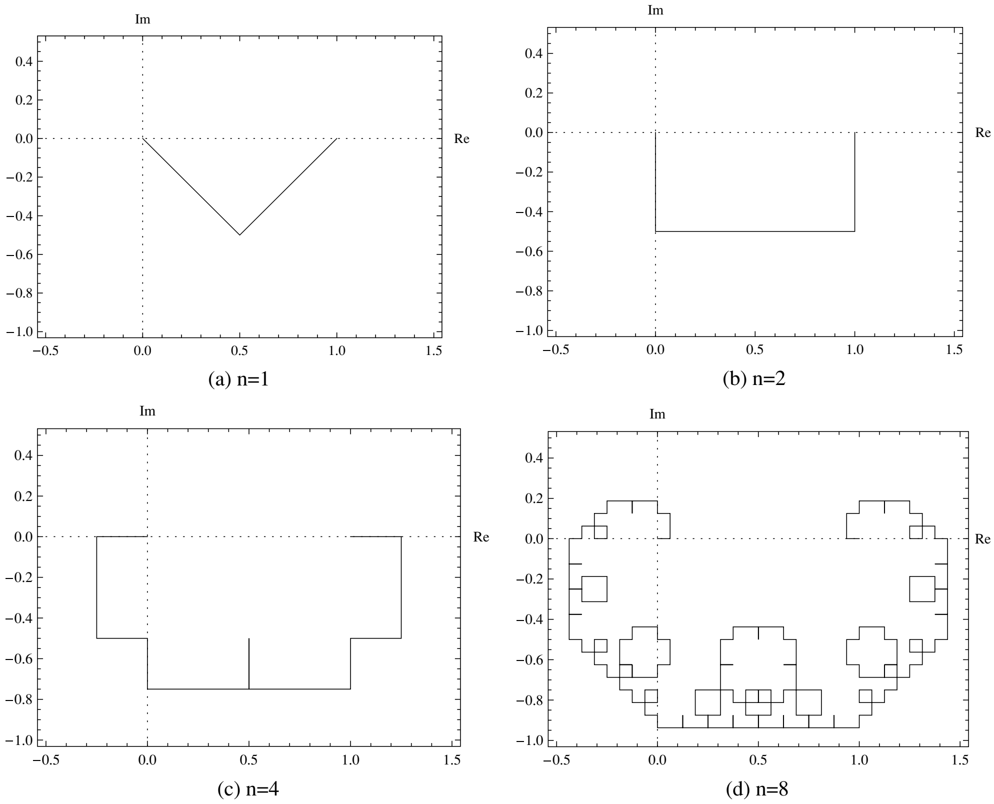
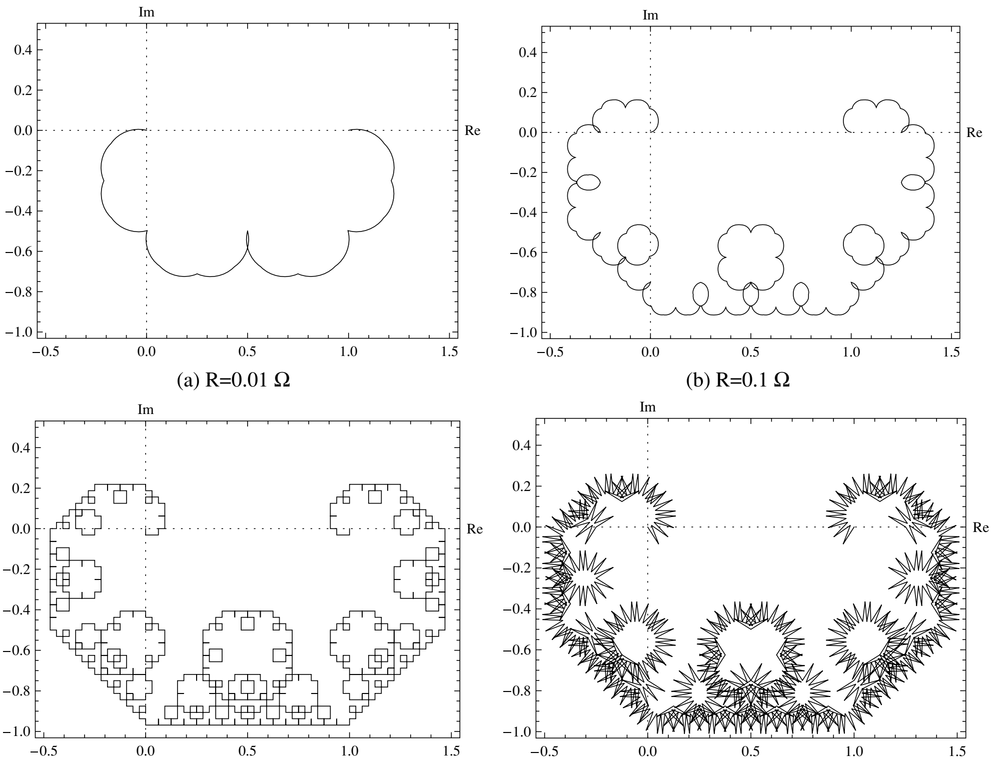


Figure 5. Voltage profile at the resistor terminals for different size ladder networks ($L = 1$ H, $C = 1$ F, $R = 1\ \Omega$, $\omega = 1$ rad s^{-1}).

shows the voltage profile for $R = 0.01\ \Omega$. The voltage magnitude in each resistor is small causing smoother voltage profile. Increasing the resistors to 0.1 Ω will change the profile as seen in figure 6(b). Figures 6(c) and (d) show the voltage profile of the resistors connected at the boundary for 1 Ω and for 10 Ω, respectively. When the values of the resistors increase, the voltage magnitudes across the resistors also increase. But, the power dissipation in each resistor does not change and it is equal to approximately 0.976 563 mW. For the total of 1024 resistors, power dissipation adds up to 1 W. This result may be interesting for the ladder network because whatever the values of the resistors used, the network adjusts the voltage across resistors so that the power dissipation stays constant. Changing the driving source frequency and/or the values of inductors and capacitors may cause different voltages across the resistors. However, the total dissipated power from the network will be equal to the input power from the source in steady state signifying that the network does not violate the first law of thermodynamic as one might expect.

Interestingly, the voltage profile across the resistors shown in figures 5 and 6(c) can also be developed without solving the algebraic voltage–current equations. Fractal-like forms similar to figures 5 and 6(c) can be described with L-systems (Lindenmayer systems, [7]). L-systems are a mathematical formulation proposed by Aristed Lindenmayer in 1968 for the study of theory of biological development. Recently, L-systems have been used for the generation of fractals and realistic modelling of plants [8]. The recursive nature of the L-systems' rules



(a) R=0.01 Ω (b) R=0.1 Ω (c) R=1 Ω (d) R=10 Ω

Figure 6. Voltage profile at the resistor terminals for $n = 10$ and $L = 1$ H, $C = 1$ F, $\omega = 1$ rad s^{-1}.

leads to self-similarity. For example, figure 5(a) can be created by replacing the straight line by the other two sides of a right-angled isosceles triangle built on it. Applying the same rule to the each line segment created will lead to figure 5(b), applying the rule four times to figure 5(c), eight times to figure 5(d), and ten times to figure 6(c). The fractal image created by this rule is actually called the Lévy C curve or Lévy dragon [9]. The Lévy dragon is a well-known fractal introduced by Paul Lévy in 1938. In the Lévy dragon, line segments in each stage will be smaller than the original (previous) by a factor of $\sqrt{2}$. The length of the lines will increase by $2/\sqrt{2}$ from the previous stage and therefore, the total length of the line after n stages will be equal to $(2/\sqrt{2})^n$. Duvall and Keesling proved that the Hausdorff dimension of the boundary of the Lévy fractal is rigorously greater than one and they estimated the dimension as 1.934 007 183 [10]. A detailed study of general families of self-similar curves of the Lévy dragon can be found in [11].

5. Conclusion

A study of a special ladder network constructed by an elementary two-terminal network consisting of a parallel resistor–inductor block in series with a parallel resistor–capacitor block is done to reveal voltage profile and dissipated power of the connected resistors. By using the nodal admittance approach, it is shown numerically that for steady state, the dissipated power from the resistors approximates to $\sqrt{L/C}$ for sufficiently large ladder networks no matter what the values (excluding zero and infinite) of the resistors are. Therefore, non-dispersive dissipative response of the network is not true in steady state and otherwise, the ladder network will violate the first law of thermodynamics.

It is shown that the voltage profile of the resistors of the ladder network in a complex plane exhibits a fractal form. With appropriate parameters, voltage profile becomes symmetrical and reveals a well-known fractal called the Lévy C curve or Lévy dragon. The Lévy C curve is introduced by Paul Lévy in 1938. Therefore, it may be appropriate to name this special ladder network as the Lévy ladder network or circuit.

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