hops (Fig. 5) are due to feedback into the laser from the interferometer and the collimating lens. As the laser warms up the current levels at which mode hops occur changes and an adjustment of the dc current can help to move them to a point that allows for a better measurement of the frequency (Figs. 6 and 7).

The best method for accuracy and resolution to date, is one that uses a detailed signal analysis in the frequency domain. This requires powerful computational techniques and precise breakdown of the individual source characteristics. Another shortcoming of the technique is the limited operating distance due to the short coherence length of the semiconductor laser. Linewidth narrowing through the use of external cavities, or possibly, the use of gas lasers, may increase the range to as high as a kilometer.[5] If the problems discussed here can be resolved, the FMCW ranging technique shows much promise for a variety of applications and warrants further research.

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Canonical transformation to energy and "*tempus*" in classical mechanics

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In classical Hamiltonian dynamics for a system with a single degree of freedom a canonical transformation is made to new canonical variables in which the new canonical momentum is energy and its conjugate coordinate is called *tempus*. This canonical coordinate *tempus* conjugate to the energy is not necessarily the time t in which the system evolves, but is a function of the original generalized coordinate, the energy, and time t. For conservative systems *tempus* reduces to the time t, and the equations reduce to the Hamilton–Jacobi equation for Hamilton's characteristic function. For periodic or almost periodic systems, the energy and *tempus* canonical variables act as a bridge to the action and angle canonical variables. Hamilton's equations for the action and angle variables in the adiabatic limit involve a generalized Hannay (or geometrical) angle. A pendulum with a length varying in time is treated as an example.

I. INTRODUCTION

The Hamiltonian formulation of classical mechanics has the advantage that canonical transformations,[1–3] which preserve the form of Hamilton's equations, can be made from one set of canonical variables to another set. A problem which may be difficult to solve in terms of the original canonical variables may be easy to solve in terms of a new set of canonical variables. After solving the problem in terms of the new canonical variables, it is necessary to transform back to the original coordinates to obtain the solution to the original problem. Common examples of canonical transformations are the Hamilton–Jacobi theory,[4] where the new canonical variables are constant, and the action-angle variables.[5] Canonical transformations to the free particle[6] and canonical transformations to bring the Hamiltonian to a given form[7] have also been given.

In this paper a new set of canonical variables is proposed for systems with one degree of freedom. The energy E is chosen to be the new canonical momentum. The canonical coordinate conjugate to the energy is called *tempus* and denoted by T. It has the dimensions of time, but is not necessarily the time t in which the system evolves.[8,9] In general, *tempus* is a function of the old canonical coordinate q, the energy E, and time t. The advantage of using the energy E and *tempus* T as canonical variables is that a problem may be easier to solve in terms of them. Hamilton's equations for these new variables must first be solved, and then they must be transformed back to the original generalized coordinate q. The energy is also a quantity of physical interest, and this method gives the energy directly. For conservative systems Hamilton's equations for the energy and *tempus* canonical variables are the same as given by Hamilton–Jacobi theory in terms of Hamilton's characteristic function W.[4] For problems which are periodic or almost periodic, the energy and *tempus* canonical variables provide a natural bridge to the action–angle canonical variables.[5] However, energy and *tempus* canonical variables are not restricted to periodic or almost periodic systems, and can be used for other problems.

Hamilton's equations for the energy E and *tempus* T canonical variables are explicitly gauge invariant.[10] Energy is the time integral of the power transferred between the system and its environment, and is gauge invariant (up to an arbitrary additive constant).[11] On the other hand, the Hamiltonian describes the time development of the system, and is gauge dependent. Therefore, a distinction is made here between the Hamiltonian H and the energy E.[11] In some gauge the energy and the Hamiltonian are equal, but not in general.

In Sec. II the canonical transformation to energy and *tempus* canonical variables is made, and their Hamilton's equations are obtained. The method is applied in Sec. III to conservative systems and compared with Hamilton–Jacobi theory. In Sec. IV it is shown that for periodic or almost periodic systems, the energy and *tempus* canonical variables act as a bridge to the action and angle canonical variables. In the adiabatic limit Hamilton's equation for the angle variable gives a generalized Hannay (or geometrical) angle. A pendulum with a time-varying length is considered as an example in Sec. V. The conclusion is given in Sec. VI. In Appendix A the arbitrariness in the generating function is discussed. In Appendix B a proof is given of the gauge invariance of Hamilton's equations for energy and *tempus*. Appendix C discusses the action variable.

II. CANONICAL TRANSFORMATION TO ENERGY AND *TEMPUS* VARIABLES

A. Hamiltonian and energy

A system with one degree of freedom is described by a Hamiltonian $H(q,p,t)$, where p is the canonical momentum conjugate to the generalized coordinate q, and t is time. The Hamiltonian is obtained from a Lagrangian by the canonical procedure. It describes the time development of the system through Hamilton's equations

$$\dot{q}=\partial H/\partial p \tag{2.1}$$

and

$$\dot{p}=-\partial H/\partial q, \tag{2.2}$$

where the overdot denotes the total derivative with respect to time t. By eliminating the canonical momentum in Eqs. (2.1) and (2.2) the (second order) equation of motion for the system may be obtained.

On the other hand, the energy of the time-dependent system is not necessarily equal to the Hamiltonian. The energy $E=E(q,\dot{q},t)$ is a function of the generalized coordinate q, the generalized velocity $\dot{q}$, and time t. The total time derivative of the energy is the power P transferred between the system and its environment,

$$dE/dt=P. \tag{2.3}$$

From physical considerations of the problem the power transferred is known and the energy can be determined. The power, in general, is equal to the scalar product of the nonconservative force and the velocity. Examples of nonconservative systems that can be given a Hamiltonian formulation are (1) a charged particle in a time-dependent electromagnetic field,[11] and (2) a damped harmonic oscillator. In each case the energy is the sum of the kinetic energy and the conservative potential energy, and depends on the coordinates, velocities, and (possibly) the time. On the other hand, the Hamiltonian is obtained through the canonical procedure from the Lagrangian. The Hamiltonian describes the dynamics of the system through Hamilton's equations (2.1) and (2.2). It is not gauge invariant, and consequently cannot in general be measured. Energy differences can be measured in principle, by integrating Eq. (2.3) for the power. Therefore, it is necessary to make a distinction between the energy and the Hamiltonian.[11]

B. Generating function

We shall make a canonical transformation with a generating function of the second type $S(q,p',t)$ which transforms from the old canonical variables (q,p) to a new set of canonical variables (q',p').[1] We choose the new canonical momentum p' to be the energy $p'=E$. The new generalized coordinate q' conjugate to it is called the canonical variable *tempus* $q'=T$. The canonical variable *tempus* T conjugate to E is in general different from the time t in terms of which the system evolves. Because the transformation is canonical, the generating function of the second type $S(q,E,t)$ satisfies[1,10]

$$p=(\partial S/\partial q)_{E,t}, \tag{2.4}$$

and the canonical coordinate *tempus* is obtained from[1,10]

$$T=(\partial S/\partial E)_{q,t}. \tag{2.5}$$

The generating function S can be obtained by integrating Eq. (2.4), which gives

$$S(q,E,t)=\int_0^q p(\bar{q},E,t)d\bar{q}+S(0,E,t), \tag{2.6}$$

where we need the old canonical momentum p as a function of q, E, and t. The function $S(0,E,t)$ is arbitrary, but does not change the dynamics. It can often be chosen to be zero for convenience. The role of $S(0,E,t)$ in Eq. (2.6) is discussed in Appendix A. From Hamilton's equation in Eq. (2.1) we have $\dot{q}=\dot{q}(q,p,t)$, and so the energy is $E=E(q,\dot{q},t)=E(q,\dot{q}(q,p,t),t)=E(q,p,t)$, the latter being of course a different function than $E(q,\dot{q},t)$. Since canonical transformations must be invertible, we *assume* we can solve the energy equation for $p=p(q,E,t)$ and use this expression in Eq. (2.6) to obtain the generating function $S(q,E,t)$.

The *tempus* variable $T=T(q,E,t)$ is obtained from Eq. (2.5) by partial differentiation of Eq. (2.6) with respect to the energy. We *assume* T can be solved for the generalized coordinate $q=q(T,E,t)$, which gives the solution to our problem if T and E are known as functions of time t. Since T and E are canonical variables, they satisfy Hamilton's equations.

C. Hamilton's equations for energy and *tempus* variables

Hamilton's equations for the canonical variables $(q',p')=(T,E)$ have the same form as Eqs. (2.1) and (2.2), viz.,

$$\dot{T}=(\partial H'/\partial E)_{T,t}, \tag{2.7}$$

and

$$\dot{E}=-(\partial H'/\partial T)_{E,t}. \tag{2.8}$$

The new Hamiltonian H' is[1–3]

$$H'=H+(\partial S/\partial t)_{q,E}. \tag{2.9}$$

The energy E and the Hamiltonian H are not necessarily equal, so we define Φ to be their difference,[11]

$$\Phi = H - E, \tag{2.10}$$

where the energy $E=E(q,\dot{q},t)$, $H=H(q,p,t)$, and $\dot{q}$ and p are related by Hamilton's equation (2.1). The function Φ can thus be expressed in terms of q, p, and t. When Eq. (2.10) for H is used in Eq. (2.9), and the new Hamiltonian is substituted into Eqs. (2.7) and (2.8), the result is

$$\dot{T} = 1 + [\partial(\Phi + \partial S/\partial t)/\partial E]_{T,t} \tag{2.11}$$

and

$$\dot{E} = -[\partial(\Phi + \partial S/\partial t)/\partial T]_{E,t}, \tag{2.12}$$

respectively. The function $\Phi + \partial S/\partial t$ must be expressed in terms of E and T from Eqs. (2.4) and (2.5) before the partial differentiation with respect to E and T in Eqs. (2.11) and (2.12), respectively, can be performed. Because E and T are canonical variables, $\partial E/\partial T = 0$. From Eq. (2.3) the right-hand side of Eq. (2.12) is equal to the power.

Equations (2.11) and (2.12) are shown to be invariant under gauge transformations in Appendix B.

III. CONSERVATIVE SYSTEMS

We apply the formulation of Sec. II here to conservative systems and show its relationship to Hamilton–Jacobi theory.

A. Energy and *tempus* canonical variables

If the system is conservative, we can always choose a gauge such that the generating function S does not have explicit time dependence, $\partial S/\partial t = 0$, and the Hamiltonian and the energy are equal, $H=E$.[11] Hamilton's equations in Eqs. (2.11) and (2.12) then reduce to the trivial equations

$$\dot{T} = 1 \tag{3.1}$$

and

$$\dot{E} = 0, \tag{3.2}$$

respectively. The solutions to Eqs. (3.1) and (3.2) are

$$T = t - t_0, \tag{3.3}$$

and

$$E = E_0, \tag{3.4}$$

where t_0 and E_0 are arbitrary constants. Equation (3.3) shows that in this case the canonical variable *tempus* T reduces to the time difference $t - t_0$.

The solution to the original problem is obtained by solving $T = T(q,E)$ in Eq. (2.5) for q, which gives $q = q(E,T)$. For conservative systems, $q = q(E_0, t-t_0) = q(t)$ is the solution for given values of the constants E_0 and t_0.

B. Hamilton–Jacobi theory

For conservative systems, Hamilton–Jacobi theory[4] reduces to this formulation in terms of energy and *tempus* canonical variables. In Hamilton–Jacobi theory, a canonical transformation is made from the canonical variables (q,p) to the new canonical variables (q_{HJ}, p_{HJ}) using a generating function $S_{HJ}(q,p_{HJ},t)$ such that the new Hamiltonian $H_{HJ} = 0$.

If the Hamiltonian is time independent, then the generating function for Hamilton–Jacobi theory may be written as[4]

$$S_{HJ}(q,p_{HJ},t) = W(q,p_{HJ}) - p_{HJ}t, \tag{3.5}$$

where $W(q,p_{HJ})$ is Hamilton's characteristic function. The old canonical momentum is

$$p = \partial S_{HJ}/\partial q = \partial W/\partial q \tag{3.6}$$

and the new coordinate is

$$q_{HJ} = \partial S_{HJ}/\partial p_{HJ} = \partial W/\partial p_{HJ} - t. \tag{3.7}$$

Because the new Hamiltonian $H_{HJ} = 0$, Hamilton's equations give the result that q_{HJ} and p_{HJ} are constants. If we choose for these constants

$$p_{HJ} = E, \quad q_{HJ} = -t_0 \tag{3.8}$$

then Eq. (3.7) becomes

$$\partial W/\partial E = t - t_0. \tag{3.9}$$

For conservative systems Eqs. (2.4) and (2.5) become Eqs. (3.6) and (3.9), respectively, if the generating function $S(q,E) = W(q,p_{HJ})$, Hamilton's characteristic function. The T in Eq. (2.5) for a conservative system is $t - t_0$ by Eq. (3.3). The generating function S in Eq. (2.6) is a solution of the Hamilton–Jacobi equation for W. Therefore, for conservative systems the energy and *tempus* canonical formulation is equivalent to the Hamilton–Jacobi theory for Hamilton's characteristic function.[12] For energy and *tempus* variables the equations are obtained directly, while the Hamilton–Jacobi theory requires the separation of variables in Eq. (3.5).

IV. ACTION AND ANGLE VARIABLES

For problems which are periodic or almost periodic, action and angle variables can be used for simplification.[5] For these problems, the energy and *tempus* canonical variables provide a bridge to the action and angle variables.

A. Transformation to action and angle variables

Some problems may be characterized by a (possibly time dependent) angular frequency $\omega = \omega(t)$. In these cases the energy E and *tempus* T variables may be expressed in terms of the angle θ and the action J variables as (see Appendix C)

$$E = \omega J, \quad T = \theta/\omega. \tag{4.1}$$

When Eq. (4.1) is substituted into Eqs. (2.11) and (2.12), the result is

$$\dot{\theta} = (\dot{\omega}/\omega)\theta + \omega + \omega\{\partial[\Phi + (\partial S/\partial t)_{q,E}]/\partial E\}_{T,t} \tag{4.2}$$

and

$$\dot{J} = -(\dot{\omega}/\omega)J - \omega^{-1}\{\partial[\Phi + (\partial S/\partial t)_{q,E}]/\partial T\}_{E,t}, \tag{4.3}$$

respectively. A new generating function $S_A(q,J,t)$ in terms of the action (hence the subscript A) may be defined as being equal to the generating function $S(q,E,t)$,

$$S_A(q,J,t) = S_A(q,E/\omega,t) = S(q,E,t). \tag{4.4}$$

In Eqs. (4.2) and (4.3) the partial derivative of S with respect to t at constant q and E can be expressed in terms of the generating function S_A as

$$(\partial S/\partial t)_{q,E}=[\partial S_A(q,E/\omega,t)/\partial t]_{q,E}$$
$$=(\partial S_A/\partial J)_{q,t}[\partial(E/\omega)/\partial t]_E+(\partial S_A/\partial t)_{q,J}$$
$$=\theta E(-\dot{\omega}/\omega^2)+(\partial S_A/\partial t)_{q,J}. \quad (4.5)$$

Since $S_A(q,J,t)$ is the generating function for the new generalized coordinate θ and the new canonical momentum J, we have

$$\theta=(\partial S_A/\partial J)_{q,t}, \quad (4.6)$$

which can be obtained from Eq. (2.5) by multiplying by ω. Equation (4.6) is used in Eq. (4.5) to obtain the final line. When Eqs. (4.1) and (4.5) are used in Hamilton's equations (4.2) and (4.3), we obtain

$$\dot{\theta}=\omega+\{\partial[\Phi+(\partial S_A/\partial t)_{q,J}]/\partial J\}_{\theta,t} \quad (4.7)$$

and

$$\dot{J}=-\{\partial[\Phi+(\partial S_A/\partial t)_{q,J}]/\partial\theta\}_{J,t}, \quad (4.8)$$

respectively, where $\omega=\omega(t)$ may be a function of time. The partial derivative of S_A with respect to time is taken at constant q and J. The first term on the right-hand side of Eqs. (4.2) and (4.3) is canceled by the appropriate derivatives of the first term on the right-hand side of Eq. (4.5). Equations (4.7) and (4.8) are Hamilton's equations for the angle and action variables, respectively.

B. Problems with no explicit time dependence in S_A

In the class of problems where the time dependence is only in the frequency, Hamilton's equations (4.7) and (4.8) may easily be solved. In these problems $S(q,E,t)=S_A(q,E/\omega)$, so that $\partial S_A/\partial t=0$ at constant q and J. If, in addition, the Hamiltonian is also the energy, $H=E$, then $\Phi=0$ from Eq. (2.10). Therefore, Hamilton's equations in Eqs. (4.7) and (4.8) become

$$\dot{\theta}=\omega(t) \quad (4.9)$$

and

$$\dot{J}=0, \quad (4.10)$$

respectively. Integrating Eq. (4.9), we obtain

$$\theta(t)=\int_0^t dt'\,\omega(t')+\theta_0, \quad (4.11)$$

where $\theta_0=\theta(0)$. If ω is constant, Eq. (4.11) becomes $\theta=\omega t+\theta_0$. Integrating Eq. (4.10), we obtain

$$J=J_0, \quad (4.12)$$

a constant. From Eq. (4.6) $\theta=\theta(q,J)$, so we can solve for q in terms of θ and J. The solution to our original problem is $q=q(\theta,J_0)=q(t)$, using Eqs. (4.11) and (4.12).

C. Adiabatic approximation

In general for periodic (or almost periodic) systems Hamilton's equations (4.7) and (4.8) may be difficult to solve exactly. They can be rewritten by adding and subtracting the angular average of the last term on the right-hand side,

$$\dot{\theta}=\omega+\langle\partial(\Phi+\partial S_A/\partial t)/\partial J\rangle+[\partial(\Phi+\partial S_A/\partial t)/\partial J$$
$$-\langle\partial(\Phi+\partial S_A/\partial t)/\partial J\rangle] \quad (4.13)$$

and

$$\dot{J}=-\langle\partial(\Phi+\partial S_A/\partial t)/\partial\theta\rangle-[\partial(\Phi+\partial S_A/\partial t)/\partial\theta$$
$$-\langle\partial(\Phi+\partial S_A/\partial t)/\partial\theta\rangle], \quad (4.14)$$

respectively, where the angular average is defined as

$$\langle\cdots\rangle=(2\pi)^{-1}\int_0^{2\pi} d\theta\cdots. \quad (4.15)$$

In the adiabatic approximation the term in the brackets in Eqs. (4.13) and (4.14) can be neglected,[13] and we have

$$\dot{\theta}=\omega(t)+\langle\partial(\Phi+\partial S_A/\partial t)/\partial J\rangle \quad (4.16)$$

and

$$\dot{J}=0, \quad (4.17)$$

since $\langle\partial(\Phi+\partial S_A/\partial t)/\partial\theta\rangle=0$. From Eq. (4.17) J is an adiabatic invariant. Equation (4.16) can be integrated to give

$$\theta(t)=\int_0^t dt'\,\omega(t')+\Delta\theta_H(t). \quad (4.18)$$

The first term on the right-hand side of Eq. (4.18) is called the dynamical angle and the second term is a generalized Hannay angle[14]

$$\Delta\theta_H(t)=\int_0^t dt'\langle\partial(\Phi+\partial S_A/\partial t')/\partial J\rangle. \quad (4.19)$$

In Appendix B it is shown that the generalized Hannay angle is invariant under gauge transformations. When $\Phi=0$ and the adiabatic time-dependent parameters are periodic with a period t, Eq. (4.19) reduces to the usual Hannay angle.[15–17]

V. PENDULUM WITH VARIABLE LENGTH

The method developed in Sec. II is illustrated by considering a pendulum with a length varying in time.[18]

A. Generating function

The Hamiltonian H for a pendulum with a small angular displacement is

$$H=p^2/2ml^2+\tfrac{1}{2}mglq^2=E, \quad (5.1)$$

where p is the canonical momentum (angular momentum) conjugate to the generalized coordinate (angle) q. The mass of the bob is m, the acceleration due to gravity is g, and the length $l=l(t)$ of the pendulum varies in time. The energy E is equal to the Hamiltonian, $H=E$, so from Eq. (5.1) the canonical momentum $p=p(q,E,t)$ is

$$p=[2ml^2(E-\tfrac{1}{2}mglq^2)]^{1/2}. \quad (5.2)$$

When Eq. (5.2) is substituted into Eq. (2.6) and integrated, we obtain

$$S(q,E,t)=\tfrac{1}{2}ql[2m(E-\tfrac{1}{2}mglq^2)]^{1/2}$$
$$+(l/g)^{1/2}E\sin^{-1}[(mgl/2E)^{1/2}q], \quad (5.3)$$

where we choose $S(0,E,t)=0$. From Eq. (2.5) the canonical variable *tempus* T is obtained by differentiating Eq. (5.3) with respect to E, which gives

$$T(q,E,t)=(l/g)^{1/2}\sin^{-1}[(mgl/2E)^{1/2}q]. \quad (5.4)$$

When this equation is solved for the generalized coordinate q we obtain

$$q(t)=(2E/mgl)^{1/2}\sin[(g/l)^{1/2}T], \tag{5.5}$$

which gives the solution when T and E are known as functions of time.

B. Hamilton's equations

Hamilton's equations for T and E are given in Eqs. (2.11) and (2.12). In this case $\Phi=0$ because of Eqs. (5.1) and (2.10). The function $\partial S/\partial t$ is obtained by differentiating the time dependence of $l(t)$ in Eq. (5.3), which gives

$$\partial S(q,E,t)/\partial t=\tfrac{3}{4}q\dot{l}[2m(E-\tfrac{1}{2}mglq^2)]^{1/2} + \tfrac{1}{2}(lg)^{-1/2}\dot{l}E\sin^{-1}[(mgl/2E)^{1/2}q]. \tag{5.6}$$

To use Eq. (5.6) in Hamilton's equations (2.11) and (2.12) it is necessary to express q as a function of E and T. When Eq. (5.5) is substituted into Eq. (5.6), we obtain

$$\partial S/\partial t=-(3/2)(d\ln\omega/dt)\omega^{-1}E\sin(2\omega T) -(d\ln\omega/dt)ET, \tag{5.7}$$

where the time-dependent angular frequency is $\omega=(g/l)^{1/2}$.

When Eq. (5.7) is used in Hamilton's equation for T in Eq. (2.11), the result is

$$\dot{T}=1-(d\ln\omega/dt)T-(3/2)(d\ln\omega/dt)\omega^{-1}\sin(2\omega T). \tag{5.8}$$

When Eq. (5.7) is used in Hamilton's equation for E in Eq. (2.12), the result is

$$\dot{E}=(d\ln\omega/dt)E+3(d\ln\omega/dt)E\cos(2\omega T). \tag{5.9}$$

Dividing Eq. (5.9) by E, we obtain

$$d\ln(E/\omega)/dt=3(d\ln\omega/dt)\cos(2\omega T). \tag{5.10}$$

It is natural now to use the action and angle variables,

$$J=E/\omega,\quad \theta=\omega T, \tag{5.11}$$

from Eq. (4.1). In terms of these variables Eq. (5.10) becomes

$$d\ln J/dt=3(d\ln\omega/dt)\cos(2\theta). \tag{5.12}$$

If Eq. (5.8) is multiplied by ω, we obtain the equation for the angle variable

$$\dot{\theta}=\omega-(3/2)(d\ln\omega/dt)\sin(2\theta). \tag{5.13}$$

Equations (5.12) and (5.13) are simpler than the corresponding Eqs. (5.9) and (5.8), respectively.

C. Adiabatic limit

In the adiabatic limit, the frequency ω is slowly varying so $d\ln\omega/dt$ is small. If we average the right-hand sides of Eqs. (5.12) and (5.13) over the angle θ from 0 to 2π as in Eq. (4.15), we have

$$\dot{J}=0 \tag{5.14}$$

and

$$\dot{\theta}=\omega, \tag{5.15}$$

respectively. Equation (5.14) shows that J=constant, so it is an adiabatic invariant. The solution to Eq. (5.15) is

$$\theta(t)=\int_0^t dt'\,\omega(t'), \tag{5.16}$$

which is the dynamical angle. The Hannay angle in Eq. (4.19) is zero in this case. When Eq. (5.11) is used in Eq. (5.5) we have the solution $q(t)$ in terms of the angle $\theta(t)$,

$$q(t)=q_{\max}(t)\sin\theta(t), \tag{5.17}$$

where the amplitude is

$$q_{\max}(t)=(4J^2/m^2gl^3)^{1/4}. \tag{5.18}$$

Since J is an adiabatic invariant by Eq. (5.14), the amplitude can be written in terms of the length as[18]

$$q_{\max}(t)=[l(0)/l(t)]^{3/4}q_{\max}(0), \tag{5.19}$$

which gives the dependence of the amplitude of the pendulum in terms of its length $l(t)$.

VI. CONCLUSION

We make a canonical transformation from old canonical variables (q,p) to new canonical variables (q',p'), where the new canonical momentum $p'=E$ is the energy and the new canonical coordinate $q'=T$ is called *tempus*. The coordinate *tempus* T conjugate to the energy E has dimensions of time, but it is in general a function of q, E, and time t. A problem may be easier to solve in terms of the new canonical variables and then transformed back to give the original coordinate $q(t)$.[1–3] The energy of the system may be of interest in its own right, and the solution of Hamilton's equations for the new variables gives the energy directly. For conservative systems, the formulation in terms of energy and *tempus* canonical variables gives the Hamilton–Jacobi equations in terms of Hamilton's characteristic function W.[4] For systems which are periodic or almost periodic, the energy and *tempus* canonical variables provide a bridge to the action and angle variables,[5] respectively. Hamilton's equations for the angle variable in the adiabatic limit involves a generalized Hannay (or geometrical) angle.[15–17] A pendulum whose length depends on the time is used to illustrate how the energy and *tempus* canonical variables provide a link to the action and angle canonical variables in terms of which the problem is simplified.[13]

Approximation methods, like the adiabatic approximation used here, may be simpler or more accurate when applied to Hamilton's equations for energy and *tempus*. Energy and *tempus* canonical variables may be applied to other systems, not only periodic or almost periodic systems. Nonconservative systems can be treated as long as they can be given a Hamiltonian formulation.[19,20] The method can perhaps be generalized to apply to one degree of freedom in a system with many degrees of freedom.

Energy and *tempus* as canonical variables is another technique for dealing with classical Hamiltonian problems. The method can be applied to conservative systems and to systems which are periodic (or almost periodic), where it gives standard methods for dealing with these systems. An advantage of the method is that it may be applied to any one dimensional problem, as long as the required equations are invertible. I am looking for problems that may be more easily solved by using this method than by other methods.

Apart from applications, however, it is instructive to have another choice of canonical variables when teaching about canonical transformations in classical mechanics.

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APPENDIX A: ARBITRARINESS IN THE GENERATING FUNCTION

In Eq. (2.6) the function $S(0,E,t)$ is arbitrary, since it is the integration "constant" after integrating Eq. (2.4). If the function $S(0,E,t)$ is changed to a new function $\tilde{S}(0,E,t)$ Hamilton's equations are form invariant, because we are making a canonical transformation.

The generating function in Eq. (2.6) can be changed to

$$\tilde{S}(q,E,t)=S(q,E,t)+\Omega(E,t), \tag{A1}$$

where $\Omega(E,t)=\tilde{S}(0,E,t)-S(0,E,t)$. Equation (A1) is similar to Eq. (B5), except that Ω is a function of the energy E and time t, while Λ is a function of the generalized coordinate q and time t. The new generalized coordinate *tempus* $\tilde{T}$ obtained from Eq. (2.5) is

$$\tilde{T}(q,E,t)=(\partial\tilde{S}/\partial E)_{q,t}=T+\partial\Omega/\partial E. \tag{A2}$$

Equation (A2) may be inverted to obtain $q=q(\tilde{T},E,t)$, which solves the problem for $q(t)$ if $\tilde{T}$ and E are known functions of the time. The energy E, which depends only on q, $\dot{q}$ and t, is invariant under the change in the generating function in Eq. (A1).

Hamilton's equations (2.11) and (2.12) can be written in terms of

$$G=\Phi+\partial S/\partial t. \tag{A3}$$

After changing the generating function as in Eq. (A1) we have a new $\tilde{G}$ defined as in Eq. (A3), which is

$$\tilde{G}=G+\partial\Omega/\partial t. \tag{A4}$$

If we change canonical variables from (T,E) to $(\tilde{T},E)$, Eq. (2.11) becomes

$$\begin{aligned}\dot{T}&=1+(\partial G/\partial E)_{\tilde{T},t}+(\partial G/\partial\tilde{T})_{E,t}(\partial^2\Omega/\partial E^2)_t\\&=1+(\partial\tilde{G}/\partial E)_{\tilde{T},t}-d(\partial\Omega/\partial E)/dt.\end{aligned} \tag{A5}$$

Equation (A5) may be written in the same form as Eq. (2.11)

$$\dot{\tilde{T}}=1+\partial(\Phi+\partial\tilde{S}/\partial t)/\partial E \tag{A6}$$

when Eq. (A2) is used. Likewise, under the same change of variables Eq. (2.12) can be written as

$$\dot{E}=-(\partial G/\partial\tilde{T})_{E,t}=-\partial(\Phi+\partial\tilde{S}/\partial t)/\partial\tilde{T}, \tag{A7}$$

in terms of the new *tempus* variable $\tilde{T}$. Equation (A7) is the same form as Eq. (2.12). From Eqs. (A6) and (A7) we see that the form of Hamilton's equations is preserved, as it should be for a canonical transformation.

The arbitrariness in $S(0,E,t)$ in Eq. (2.6) changes the generalized coordinate *tempus* T to $\tilde{T}=T+\partial\Omega/\partial E$ according to Eq. (A2), but does not change the conjugate canonical momentum, the energy E. On the other hand, the gauge transformation discussed in Appendix B changes the canonical momentum p to $p''=p+\partial\Lambda/\partial q$ according to Eq. (B2), but does not change the generalized coordinate $q''=q$, according to Eq. (B3). The distinction between a gauge transformation and the arbitrariness in the generating function is that q and E are interchanged.

APPENDIX B: GAUGE INVARIANCE OF HAMILTON'S EQUATIONS FOR ENERGY AND *TEMPUS*

In classical mechanics a gauge transformation is a canonical transformation[10] from the old canonical variables (q,p) to the new canonical variables (q'',p'') with a generating function of the second type[1]

$$F_2(q,p'',t)=qp''-\Lambda(q,t), \tag{B1}$$

where $\Lambda(q,t)$ is an arbitrary differentiable function of q and t. For Eq. (B1) to be a generating function of the second type, it must satisfy

$$p=\partial F_2/\partial q=p''-\partial\Lambda(q,t)/\partial q \tag{B2}$$

and

$$q''=\partial F_2/\partial p''=q. \tag{B3}$$

By Eq. (B2) the new canonical momentum $p''=p+\partial\Lambda/\partial q$ is shifted from the old canonical momentum p. By Eq. (B3) the generalized coordinate q is unchanged. The new Hamiltonian H'' for the new canonical variables (q'',p'') is

$$H''=H+\partial F_2/\partial t=H-\partial\Lambda/\partial t. \tag{B4}$$

We now make another canonical transformation to the energy E and *tempus* T canonical variables. The energy $E=E(q,\dot{q},t)$ is gauge invariant (up to a constant) because it depends only on the coordinate q, the velocity $\dot{q}$, and time t. From Hamilton's equation $\dot{q}=\dot{q}(q,p'',t)$, so $E=E(q,\dot{q},t)=E[q,\dot{q}(q,p'',t),t]=E(q,p'',t)$. We can solve $E=E(q,p'',t)$ for $p''=p''(q,E,t)$ and obtain the generating function S'' from Eq. (2.6) as

$$\begin{aligned}S''(q,E,t)&=\int_0^q p''(\bar{q},E,t)d\bar{q}+S''(0,E,t)\\&=S(q,E,t)+\Lambda(q,t),\end{aligned} \tag{B5}$$

where we use Eq. (B2) and choose $S''(0,E,t)=S(0,E,t)+\Lambda(0,t)$. From Eq. (2.5) the new *tempus* variable $T''=\partial S''/\partial E=\partial S/\partial E=T$ from Eq. (B5). Hence both E and T are gauge invariant.

The difference Φ'' in Eq. (2.10) between the new Hamiltonian H'' in Eq. (B4) and the energy E is

$$\Phi''=H''-E=\Phi-\partial\Lambda/\partial t. \tag{B6}$$

Hamilton's equations for the canonical variables E and T have the same form as Eqs. (2.11) and (2.12),

$$\dot{T}=1+\partial(\Phi''+\partial S''/\partial t)/\partial E \tag{B7}$$

and

$$\dot{E}=-\partial(\Phi''+\partial S''/\partial t)/\partial T, \tag{B8}$$

respectively. The function $\Phi''+\partial S''/\partial t$ is, however, gauge invariant

$$\Phi''+\partial S''/\partial t=(\Phi-\partial\Lambda/\partial t)+\partial(S+\Lambda)/\partial t$$
$$=\Phi+\partial S/\partial t, \tag{B9}$$

from Eqs. (B5) and (B6). Therefore, Eqs. (B7) and (B8) reduce to Eqs. (2.11) and (2.12), respectively, which proves the gauge invariance of the formulation.

The Hannay angle $\Delta\theta_H(t)$ in Eq. (4.19) is also gauge invariant. From Eq. (4.5) we have

$$\Phi+(\partial S/\partial t)_{q,E}=\Phi+(\partial S_A/\partial t)_{q,J}-\theta E\dot{\omega}/\omega^2. \tag{B10}$$

From Eq. (B9) the left-hand side of Eq. (B10) is gauge invariant, and θ, E, and ω are also gauge invariant, so the function $\Phi+\partial S_A/\partial t$, which appears in Eq. (4.19), is gauge invariant.[21]

APPENDIX C: ACTION VARIABLE J

We have taken Eq. (4.1) to be the definition of the action J and angle θ canonical variables. The action variable is defined differently in Ref. 1 (p. 460). Here we show that for one degree of freedom the two definitions are equivalent.

When the generating function $S(q,E,t)$ in Eq. (2.6) is used in the definition of the canonical coordinate *tempus* in Eq. (2.5), we obtain

$$T=\frac{\partial}{\partial E}\int_0^q p(\bar{q},E,t)d\bar{q}+T_0, \tag{C1}$$

where $T_0=\partial S(0,E,t)/\partial E$. When Eq. (C1) is multiplied by the angular frequency ω, and Eq. (4.1) is used, we have

$$\theta-\theta_0=\omega\frac{\partial}{\partial E}\int_0^q p(\bar{q},E,t)d\bar{q}, \tag{C2}$$

where $\theta_0=\omega T_0$. For a complete cycle $\theta-\theta_0=2\pi$ and Eq. (C2) becomes

$$2\pi=\omega\frac{\partial}{\partial E}\oint p(q,E,t)dq. \tag{C3}$$

When Eq. (C3) is integrated with respect to the energy E, we obtain

$$E=\omega(2\pi)^{-1}\oint p\,dq, \tag{C4}$$

where the (possibly time dependent) integration "constant" is chosen to be zero. When Eq. (C4) is compared with Eq. (4.1), we see that the action J is

$$J=(2\pi)^{-1}\oint p\,dq, \tag{C5}$$

which agrees [up to a factor of $(2\pi)^{-1}$] with the definition in Ref. 1 (p. 460).

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[4] See Ref. 1, Chap. 10; Ref. 2, Chap. VIII; Ref. 3, pp. 789–852.

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THE STORY OF TWENTIETH-CENTURY SCIENCE

Twentieth-century science has a grand and impressive story to tell. Anyone framing a view of the world has to take account of what it has to say ... It is a non-trivial fact about the world that we can understand it and that mathematics provides the perfect language for physical science: that, in a word, science is possible at all.

J. C. Polkinghorne (1983), quoted in Andrew Pickering, *Constructing Quarks—A Sociological History of Particle Physics* (University of Chicago, Chicago, 1984), p. 413.