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Phase Transition in the Number Partitioning Problem

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Number partitioning is an *NP*-complete problem of combinatorial optimization. A statistical mechanics analysis reveals the existence of a phase transition that separates the easy- from the hard-to-solve instances and that reflects the pseudopolynomiality of number partitioning. The phase diagram and the value of the typical ground-state energy are calculated. [S0031-9007(98)07670-4]

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Computer science has recently discovered the notion of phase transition in random combinatorial problems and its possible connections with algorithmic complexity. In such a context, statistical physics may provide an interesting perspective for understanding problems in theoretical computer science. In this Letter we calculate the statistical mechanics of one of the core problems in theoretical computer science.

The number partitioning problem is an easily formulated optimization problem: Given a set $\mathcal{A} = \{a_1, a_2, \ldots, a_N\}$ of positive numbers, find a partition, i.e., a subset $\mathcal{A}' \subset \mathcal{A}$, such that the residue

$$E = \left| \sum_{a_j \in \mathcal{A}'} a_j - \sum_{a_j \notin \mathcal{A}'} a_j \right| \tag{1}$$

is minimized. A partition with $E = 0$ is called *perfect.* The decision variant of the number partitioning problem is to determine if there is a perfect partition or not.

Number partitioning is of both theoretical and practical importance. It is one of Garey and Johnson's six basic *NP*-complete problems that lie at the heart of the theory of *NP* completeness [1]. Among the many practical applications one finds multiprocessor scheduling and the minimization of VLSI circuit size and delay.

A partition can be encoded by numbers $s_j = \pm 1$: $s_j = 1$ if $a_j \in \mathcal{A}'$, $s_j = -1$ otherwise. The cost function then reads

$$E = \left| \sum_{j=1}^{N} a_j s_j \right|, \tag{2}$$

and the minimum partition is equivalent to the ground state of the Hamiltonian

$$H = E^2 = \sum_{i,j=1}^{N} s_i a_i a_j s_j \,. \tag{3}$$

This is an infinite range Ising spin glass with Mattis-like, antiferromagnetic couplings $J_{ij} = -a_i a_j$. The thermodynamics of this model has been investigated by Fu [2] and recently be Ferreira and Fontanari [3].

Fu claims that in the random number partitioning problem "... no phase transition of any kind is found." [2]. If Fu were right, number partitioning would be a notable exception to the observation that many *NP*-complete problems do have a phase transition, parametrized by a control parameter that separates the easy from the hard-to-solve instances [4]. For random, integer $a_i \in \{0, 1, 2, \ldots, A\}$, Gent and Walsh [5] proposed

$$\tilde{\kappa} = \frac{\log_2 A}{N} \tag{4}$$

as a control parameter: They found numerically that one typically has $O(2^N)$ perfect partitions for $\tilde{\kappa} < \kappa_c$, whereas for $\tilde{\kappa} > \kappa_c$ the number of perfect partitions drops to zero. The transition gets sharper with increasing N. Finite-size scaling leads Gent and Walsh to $\kappa_c = 0.96$ for $N \to \infty$. This result contradicts Fu's claim, but as we will see now, this type of phase transition can indeed be found in the statistical mechanics of the number partitioning problem.

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The canonical formalism of statistical mechanics requires the calculation of the partition function

$$Z = \sum_{\{S_i\}} e^{-E/T} = \sum_{\{S_i\}} \int_{-\infty}^{\infty} dx\, e^{-|x|} \delta\left(x - \frac{1}{T}\sum_{j=1}^{N} a_j s_j\right)$$
$$= 2^N \int_{-\infty}^{\infty} \frac{d\hat{x}}{2\pi} \prod_{j=1}^{N} \cos\left(\frac{a_j}{T}\,\hat{x}\right) \int_{-\infty}^{\infty} dx\, e^{-|x| + i\hat{x}x}$$
$$= 2^N \int_{-\pi/2}^{\pi/2} \frac{dy}{\pi} \prod_{j=1}^{N} \cos\left[\frac{a_j}{T}\tan(y)\right], \qquad (5)$$

where T is the temperature. We write Z as

$$Z = 2^N \int_{-\pi/2}^{\pi/2} \frac{dy}{\pi}\, e^{NG(y)}, \qquad (6)$$

with

$$G(y) = \frac{1}{N}\sum_{j=1}^{N} \ln\cos\left[\frac{a_j}{T}\tan(y)\right]. \qquad (7)$$

At this point we could use the statistical independence of the a_j and replace the sum by the average of $\ln\cos[\frac{a}{T}\tan(y)]$ over a. This is usually done in Mattis-like spin glasses [6], but we will proceed without this substitution and calculate all thermodynamic quantities as functions of $\{a_j\}$.

For large N the integral in Eq. (6) can be evaluated using the saddle-point technique. To find the saddle points of $G(y)$, we will assume that a can take on only values that are integer multiples of a fixed number Δa. For integer distributions $\Delta a = 1$, and for floating-point distributions Δa is the smallest number that can be represented with the available number of bits. This assumption leads to an infinite number of saddle points, missed in [3],

$$y_k = \arctan\left(\frac{\pi T}{\Delta a}\,k\right), \qquad k = 0, \pm 1, \pm 2, \ldots. \qquad (8)$$

The resulting series of Gaussian integrals can be evaluated as

$$Z = 2^N \sum_{k=0,\pm 1,\ldots} \int_{-\infty}^{\infty} dy\, e^{-N/2G''(y_k)y^2}$$
$$= 2^N \sqrt{\frac{2\Delta a^2}{\pi\sum_j a_j^2}}\,\coth\frac{\Delta a}{T}. \qquad (9)$$

From that we get the average energy

$$\frac{E}{T} = \frac{\Delta a}{T}\,\frac{\coth^2\frac{\Delta a}{T} - 1}{\coth\frac{\Delta a}{T}} \qquad (10)$$

and the entropy

$$S = N\ln 2 - \frac{1}{2}\ln\left(\frac{\pi\sum_j a_j^2}{2\Delta a^2}\right) + \tilde{S}\left(\frac{\Delta a}{2T}\right), \qquad (11)$$

where the thermal contribution reads

$$\tilde{S}\left(\frac{\Delta a}{T}\right) = \ln\coth\frac{\Delta a}{T} + \frac{\Delta a}{T}\,\frac{\coth^2\frac{\Delta a}{T} - 1}{\coth\frac{\Delta a}{T}}. \qquad (12)$$

Note that for finite Δa, $\tilde{S}$ vanishes at zero temperature and increases monotonically with T. The entropy can be written as

$$S = N(\kappa_c - \kappa)\ln 2 + \tilde{S}, \qquad (13)$$

with

$$\kappa_c(N) = 1 - \frac{\ln(\frac{\pi}{6}N)}{N2\ln 2} \qquad (14)$$

and

$$\kappa = \frac{\ln\frac{3}{\Delta a^2}\frac{1}{N}\sum_j a_j^2}{N2\ln 2}. \qquad (15)$$

Note that $\kappa = \tilde{\kappa} + O(\frac{1}{NA})$ for the distribution of the a_i's considered by Gent and Walsh.

For $\kappa < \kappa_c$ the entropy is extensive even for $T = 0$. According to Eq. (10), the corresponding energy is zero, hence we expect an *exponential number of perfect partitions,* in good agreement with the numerical results [5].

For $\kappa > \kappa_c$ the zero temperature entropy seems to become negative. This would be wrong because the entropy must not be smaller than ln 2 for our discrete system. To see what is going on here, note that $\kappa > \kappa_c$ means

$$2^{-N} > \Delta a\sqrt{\frac{2}{\pi\sum_j a_j^2}}, \qquad (16)$$

i.e., essentially $\Delta a = O(2^{-N})$. In this regime the contributions of $\tilde{S}$ are $O(N)$ for any finite T,

$$\tilde{S}\left(\frac{\Delta a}{T}\right) = \ln\left(\frac{T}{\Delta a}\right) + 1 + O\left(\frac{\Delta a^2}{T^2}\right), \qquad (17)$$

hence cannot be neglected. Technically we deal with this contribution by introducing an effective "zero" temperature T_0 below which the system cannot be cooled. T_0 guarantees that the contribution of $\tilde{S}$ remains $O(N)$. Its value can be calculated from the lower bound of S:

$$\ln 2 = N(\kappa_c - \kappa)\ln 2 + \tilde{S}\left(\frac{\Delta a}{T_0}\right)$$
$$\approx N(\kappa_c - \kappa)\ln 2 + \ln\left(\frac{T_0}{\Delta a}\right).$$

From that we get

$$T_0 = 2\Delta a 2^{N(\kappa - \kappa_c)} = \sqrt{2\pi\sum_j a_j^2}\,2^{-N}. \qquad (18)$$

For $\kappa > \kappa_c$ the ground-state energy E_0 reads

$$E_0 = T_0 = \sqrt{2\pi\sum_j a_j^2}\,2^{-N}. \qquad (19)$$

This equation specifies the rigorous result that the median value of E_0 is $O(\sqrt{N}\,2^{-N})$ [7].

To check Eq. (19) we consider the continuous variant of number partitioning, where the a_i are real numbers, uniformly distributed in the interval $[0, 1)$. In our formalism this means $\Delta a \to 0$ and $\sum_j a_j^2 = N/3$. We are in the $\kappa > \kappa_c$ regime and Eq. (19) becomes

$$E_0 = \sqrt{\frac{2}{3}\,\pi N}\,2^{-N} = 1.447\sqrt{N}\,2^{-N}. \qquad (20)$$

In Fig. 1, Eq. (20) is compared to numerical data. The agreement is convincing. The prefactor $\sqrt{\frac{2}{3}\pi}$ fits much better than the prefactor $\sqrt{\frac{\pi}{6e^2}} = 0.2662$, reported in [3].

To check whether $\kappa(N)$ is a control parameter with a phase transition at $\kappa_c(N)$, we did numerical simulations. For fixed N and κ we calculated the fraction of instances that have at least one perfect partition. In accordance with Gent and Walsh [5] we find that this fraction is 1 for small κ and 0 for larger κ. The transition from 1 to 0 is sharp. Figure 2 shows the numerically found transition points for $10 \le N \le 28$ compared to $\kappa_c(N)$ from Eq. (14). Again the agreement is convincing. Note that $\kappa_c(N \to \infty) = 1$. The asymptotic estimate 0.96 given by Gent and Walsh is probably due to the rather small values $N \le 30$ used in their simulations.

The two phases are very different with respect to the computational complexity of the corresponding instances. For $\kappa < \kappa_c$, a search algorithm is likely to find one of the numerous perfect partitions in short time, while the unique minimum partition for $\kappa > \kappa_c$ requires exponential time to be found. This behavior can indeed be seen in numerical experiments [5,8,9]. Referring to the solvability of the typical instance we call the two phases "easy" and "hard." In a numerical investigation, the precision of the numbers a_i is fixed and N is varied. For high precision and small values of N, we are in the "hard" phase: No perfect partition exists, and a search algorithm has to explore large parts of the configuration space. If one increases N, the search space grows exponentially, and so does the running time. On the other hand, increasing N gets us closer to the phase boundary. Beyond this threshold, the number of perfect partitions increases exponentially with N. A smart algorithm will try to find a perfect partition as early as possible. As can be seen from numerical experiments [5,8,9], this may even lead to the effect that the running time now *decreases* with increasing N. As a function of N the running time has a sharp maximum at the phase boundary: The hardest problems are those close to the threshold. A similar behavior has been found in other *NP*-complete problems like, for example, the satisfiability problem [10–12].

For bounded, integer values $0 \le a_i < A$ even the *worst case* complexity of number partitioning is polynomial in A and N [1]. This is no contradiction to the *NP* completeness since a concise encoding of an instance requires $N \log_2 A$ bits, and A is not bounded by a polynomial function of $\log_2 A$. Because of this property the number partitioning problem is called *pseudopolynomial* [1]. The exponential complexity of number partitioning relies on the fact that extremely large (or precise) input numbers a_i are allowed. This distinguishes number partitioning from many other *NP*-complete problems like, for example, the traveling salesman problem, which remains *NP* hard even if the distances are restricted to take on the values 1 and 2. Pseudopolynomiality applies if the number of bits to

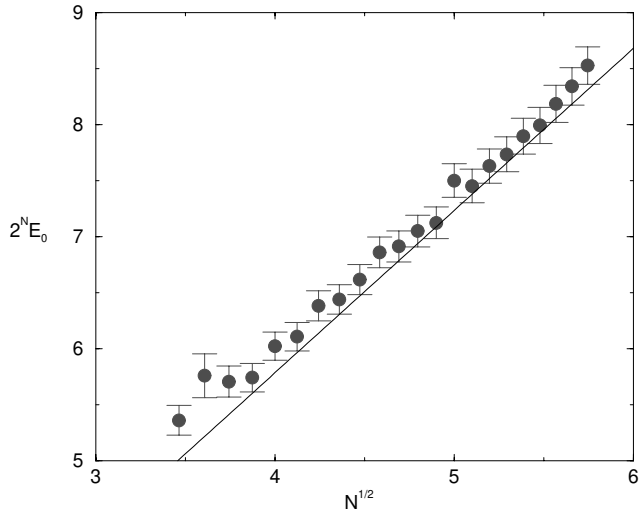

FIG. 1. Average minimum residue of the number partitioning problem with real numbers $0 \le a_i < 1$ compared to the analytical result Eq. (20) (straight line). Each data point is the average over 10^4 random samples.

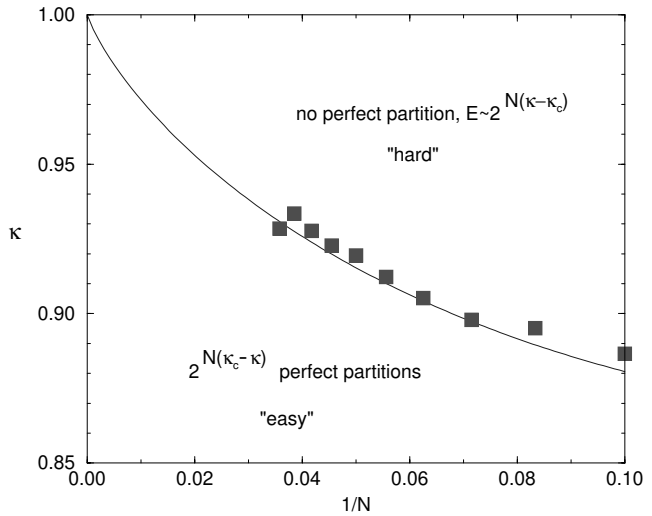

FIG. 2. Phase diagram of the random number partitioning problem. $N\kappa$ is essentially the number of bits to encode the input numbers; see Eq. (15). The squares denote the phase boundary found numerically. The solid line is given by κ_c from Eq. (14). For $\kappa < \kappa_c$, the zero temperature entropy is extensive and a search algorithm typically finds quickly one of the $O(2^N)$ perfect partitions. For $\kappa > \kappa_c$, no perfect partitions exist and the optimization problem has a hard-to-find, unique solution.

represent a_i is fixed while N increases. This means that κ decreases, i.e., we get into the easy phase for large enough N. Hence the notion easy not only refers to the *typical* (as shown here) but also to the *worst case* complexity. This is a notable feature of the number partitioning problem: The statistical mechanics results hold beyond the typical case for which they are derived.

Looking at Eqs. (13) and (19) an interpretation of the parameter κ_c suggests itself. Let the N numbers a_i each be represented by $N\kappa$ bits. Now consider the residue E bitwise: About half of all partitions will set the most significant bit of E to zero. Among those partitions, about one-half will set the second most significant bit to zero, too. Repeating this procedure we can set at most $N\kappa_c(N)$ bits to zero until running out of available partitions. If $\kappa < \kappa_c$, we get a perfect partition before reaching this point. The remaining set of available partitions has $2^{N(\kappa_c-\kappa)}$ elements. This explains the zero temperature entropy, Eq. (13). For $\kappa > \kappa_c$, the $N(\kappa > \kappa_c)$ least significant bits in E cannot be fixed by the optimum partition, leading to Eq. (19) for the residue.

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[1] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness* (W. H. Freeman, New York, 1997).
[2] Y. Fu, in *Lectures in the Sciences of Complexity,* edited by D. L. Stein (Addison-Wesley Publishing Company, Reading, Massachusetts, 1989).
[3] F. Ferreira and J. Fontanari, J. Phys. A **31**, 3417 (1998).
[4] P. Cheeseman, B. Kanefsky, and W. M. Taylor, in *Proceedings of IJCAI-91,* edited by J. Mylopoulos and R. Rediter (Morgan Kaufmann, San Mateo, CA, 1991), pp. 331–337.
[5] I. P. Gent and T. Walsh, in *Proceedings of ECAI-96,* edited by W. Wahlster (John Wiley & Sons, New York, 1996), pp. 170–174.
[6] J. Provost and G. Vallee, Phys. Rev. Lett. **50**, 598 (1983).
[7] N. Karmakar, R. M. Karp, G. S. Lueker, and A. M. Odlyzko, J. Appl. Prob. **23**, 626 (1986).
[8] R. E. Korf, in *Proceedings of the 14th IJCAI,* edited by C. S. Mellish (Morgan Kaufmann, San Mateo, CA, 1995), pp. 266–272.
[9] R. E. Korf, "A Complete Anytime Algorithm for Number Partitioning" (to be published).
[10] B. Hayes, Am. Sci. **85**, 108 (1997).
[11] R. Monasson and R. Zecchina, Phys. Rev. Lett. **76**, 3881 (1996).
[12] R. Monasson and R. Zecchina, Phys. Rev. E **56**, 1357 (1997).