Asymptotics of Lagged Fibonacci Sequences

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Consider "lagged" Fibonacci sequences $a(n) = a(n-1) + a(\lfloor n/k \rfloor)$ for $k > 1$. We show that $\lim_{n\to\infty} a(kn)/a(n) \cdot \ln n/n = k \ln k$ and we demonstrate the slow numerical convergence to this limit and how to deal with this slow convergence. We also discuss the connection between two classical results of N.G. de Bruijn and K. Mahler on the asymptotics of $a(n)$.

I. INTRODUCTION

Let $k > 1$ be an integer and consider "lagged" Fibonacci type sequences

$$a_k(n) = a_k(n-1) + a_k\left(\left\lfloor \frac{n}{k} \right\rfloor\right) \tag{1}$$

with initial value

$$a_k(0) = 1\,. \tag{2}$$

These "almost linear recurrence" has many interesting arithmetical properties [1]. The value $a_k(n)$ equals the number of k-ary partitions of kn, and the corresponding sequences are listed in the OEIS as A000123, A005704, A005705 and A005706 for $k = 2, 3, 4, 5$, respectively. In this contribution we will study the asymptotical behavior of the ratio

$$c_k(n) = \frac{a_k(kn)}{a_k(n)} \frac{\ln n}{n}\,. \tag{3}$$

The OEIS entry for A000123 quotes a conjecture due to Benoit Cloitre, claiming that

$$\lim_{n\to\infty} c_k(n) = \text{const.} = 1.63\ldots\,. \tag{4}$$

The same conjecture (but with const. $= 1.64\ldots$) appears for the related sequence A033485. We will prove that the essential part of the conjecture (existence of the limit) is true, but that its numerical part is incorrect. In particular, we will apply a classical result of de Bruijn [2] to prove that

$$\lim_{n\to\infty} c_k(n) = k \ln k\,. \tag{5}$$

Note that $2 \ln 2 = 1.386\ldots$, which differs significantly from the value in (4).

In the second part we will discuss the rate of convergence of $c_k(n)$. It turns out that this rate is so slow that straightforward numerical measurements of $c_k(n)$ cannot be used for an accurate measurement of $c_k(\infty)$. This may explain the inaccurate numerical value in (4). It turns out that another classical result on the asymptotics of $a_k(n)$ due to K. Mahler [3] can be used as a device for an accurate numerical determination of $c_k(n)$ all the way to the asymptotic regime.

In the final part we will discuss the connection between the two asymptotic formulas of de Bruijn and Mahler.

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II. ASYMPTOTICS

Using an integral representation (Mellin transformation) of the generating function for $a_k(n)$ and a saddle point integration, de Bruijn [2] showed that

$$\begin{aligned}\ln a_k(n) =& \frac{1}{2\ln k}(\ln n - \ln\ln n)^2 + \left(\frac{1}{2} + \frac{1}{\ln k} + \frac{\ln\ln k}{\ln k}\right)\ln n - \left(1 + \frac{\ln\ln k}{\ln k}\right)\ln\ln n \\ &+ \left(1 + \frac{\ln\ln k}{2\ln k}\right)\ln\ln k - \frac{1}{2}\ln(2\pi) + \psi_k\left(\log_k\left(\frac{n}{\log_k n}\right)\right) + \mathcal{O}\left(\frac{\ln^2\ln n}{\ln n}\right),\end{aligned} \tag{6}$$

where ψ_k is a periodic function with period 1,

$$\psi_k(x) = \sum_{j=-\infty}^{\infty} \alpha_j(k) \mathrm{e}^{2\pi i j x}. \tag{7}$$

The Fourier coefficients are

$$\alpha_j(k) = \frac{1}{\ln k}\,\Gamma\left(\frac{2\pi i j}{\ln k}\right)\zeta\left(1 + \frac{2\pi i j}{\ln k}\right) \qquad (j \neq 0) \tag{8}$$

and

$$\alpha_0(k) = \frac{1}{\ln k}\left(-\gamma_1 - \frac{1}{2}\gamma^2 + \frac{1}{12}\pi^2 + \frac{1}{12}\ln^2 k\right), \tag{9}$$

where $\gamma = 0.5772156649\ldots$ is the Euler constant and $\gamma_1 = -0.0728158454\ldots$ is the first Stieltjes constant.

The Fourier series for $\psi_k(x)$ converges absolutely and uniformely because the coefficients $\alpha_j(k)$ decay fast enough:

$$|\Gamma(it)| = \mathcal{O}(|t|^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}\pi|t|}) \tag{10}$$

and

$$|\zeta(1+it)| = \mathcal{O}(\ln|t|). \tag{11}$$

Plugging (6) into (3) provides us with

$$\ln c_k(n) = \ln(k\ln k) + \Delta\psi_k(n) + \mathcal{O}\left(\frac{(\ln\ln n)^2}{\ln n}\right), \tag{12}$$

where

$$\Delta\psi_k(n) = \psi_k\left(\log_k\left(\frac{n}{\log_k n}\right) - \log_k\left(1 + \frac{1}{\log_k n}\right)\right) - \psi_k\left(\log_k\left(\frac{n}{\log_k n}\right)\right). \tag{13}$$

Intuitively, $\Delta\psi_k(n)$ should vanish for $n \to \infty$, but to be sure we need to investigate the Fourier series for ψ_k in more detail. In particular, we have

$$\begin{aligned}|\Delta\psi_k(n)| &= \left|\sum_{j=-\infty}^{\infty}\alpha_j(k)\exp\left(2\pi i j\log_k\left(\frac{n}{\log_k n}\right)\right)\left(\exp\left(-2\pi i j\log_k\left(1+\frac{1}{\log_k n}\right)\right) - 1\right)\right| \\ &\le \sum_{j=-\infty}^{\infty}|\alpha_j(k)|\left|\exp\left(-2\pi i j\log_k\left(1+\frac{1}{\log_k n}\right)\right) - 1\right| \\ &\le 2\pi\sum_{j=-\infty}^{\infty}|j\,\alpha_j(k)|\left|\log_k\left(1+\frac{1}{\log_k n}\right)\right|.\end{aligned}$$

In the last line we have used the inequality

$$|\mathrm{e}^{ix} - 1| \le |x| \qquad (x \in \mathbb{R}). \tag{14}$$

Now because of (10) and (11) we know that $\sum_j |j\alpha_j(k)| < \infty$, and hence

$$\Delta\psi_k(n) = \mathcal{O}(1/\log_k n). \tag{15}$$

This concludes our proof of (5).

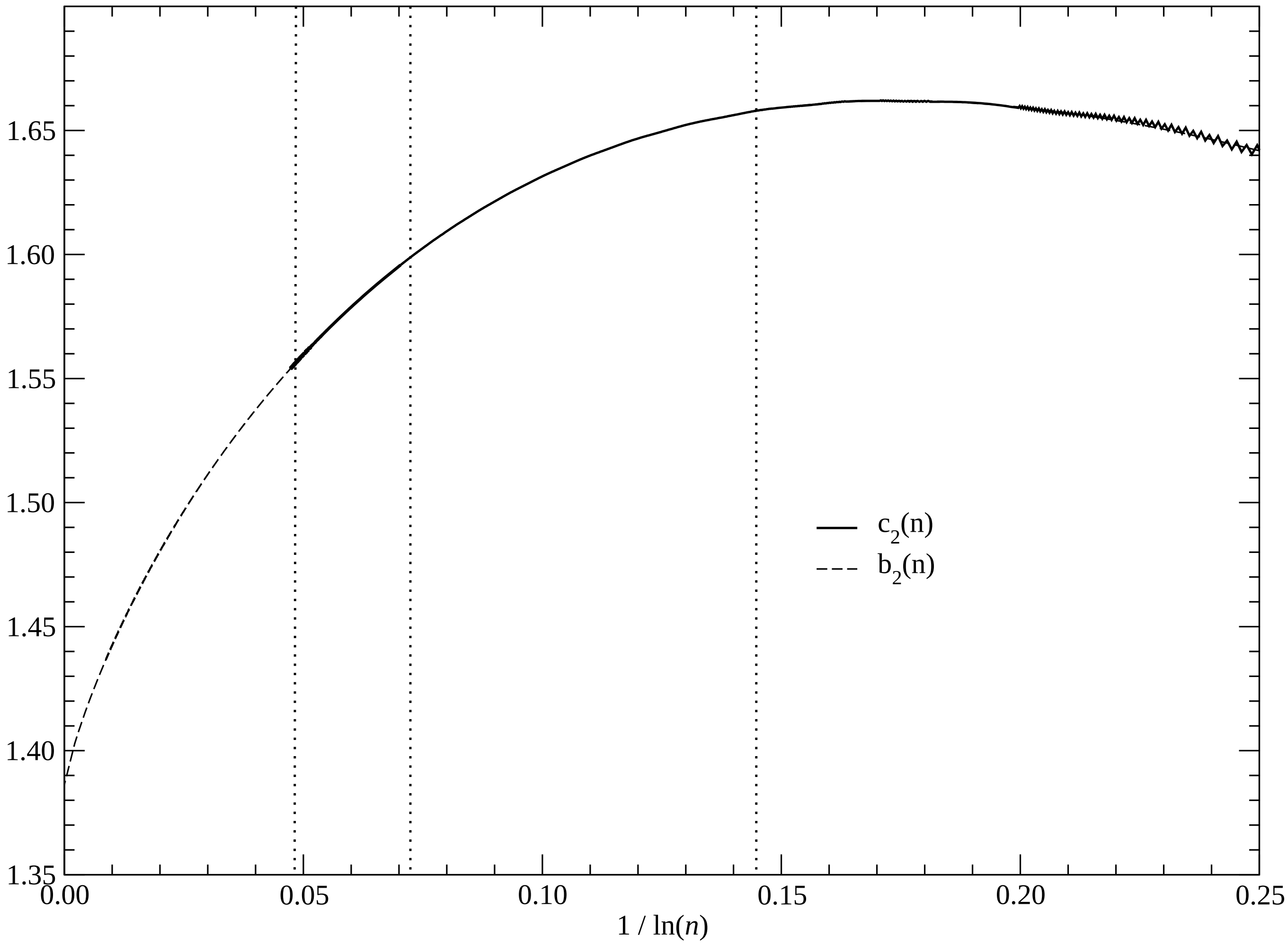

Figure 1: Numerical evaluation of $c_2(n)$ together with $b_2(n)$ from (19). The difference between c_2 and b_2 is smaller then the linewidth for all $n > 100$. The dotted vertical lines indicate $n = 10^3$, 10^6 and 10^9.

III. NUMERICAL EVALUATION

The recurrence (1) appears in the analysis of the Karmarkar-Karp differencing algorithm for number partitioning [4]. In this context we learned that the convergence to the asymptotic regime can be extremely slow. We will see that this is also true when we try to probe the asymptotics of $c_k(n)$ numerically.

To calculate $c_k(n)$, we need $a(kn)$ and $a(n)$, but because of

$$a_k(kn) = a_k(n) + k \sum_{j=0}^{n-1} a_k(j)\,, \tag{16}$$

the value of $a_k(n)$ plus the sum of the preceeding terms is sufficient. The bottleneck for calculating $a_k(n)$ is memory, not CPU time, since $n(1-1/k)$ values must be stored to compute $a_k(n)$. We used the Chinese Remainder Theorem to keep the individual numbers small and managed to calculate $c_2(n)$ for n up to $3 \cdot 10^9$ on a PC with 4 GByte of memory. As Fig. 1 shows, even these data are insufficient to extrapolate to the true asymptotic value. Numerical calculations that stop at even smaller values of n may easily misguide an extrapolation to $c_k(\infty)$.

In order to evaluate $c_k(n)$ for much larger values of n, we resort to another asymptotic result. In 1940, Mahler [3] showed that

$$a_k(n) = \mathrm{e}^{\phi_k(n)} \sum_{j=0}^{\infty} \frac{n^j}{k^{\binom{j}{2}} j!} \equiv \mathrm{e}^{\phi_k(n)}\, S_k(n) \tag{17}$$

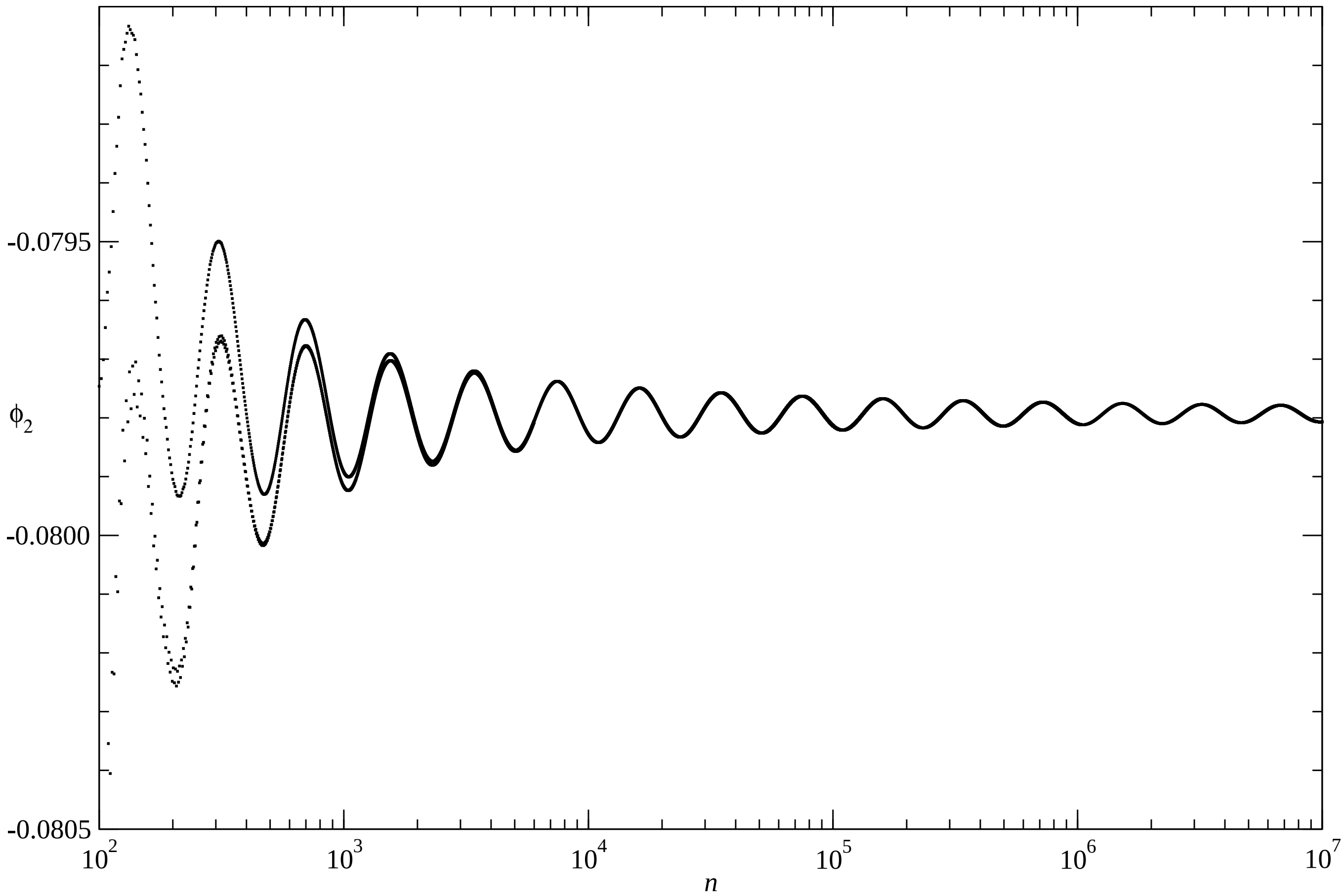

Figure 2: Numerical evaluation of ϕ_2

n	$c_2(n)$	$\frac{S_k(kn)}{S_k(n)} \cdot \frac{\ln n}{n}$
10^1	1.49668	1.50889
10^2	1.65470	1.65496
10^3	1.65791	1.65779
10^4	1.63881	1.63876
10^5	1.61782	1.61780
10^6	1.59883	1.59882
10^7	1.58237	1.58237
10^8	1.56822	1.56822
10^9	1.55600	1.55600

Table I: Exact evaluation of $c_2(n)$ versus evaluation of the series.

where $\phi_k(n) = O(1)$. The idea is to replace the numerical evaluation of $a_k(n)$ by the numerical evaluation of the sum $S_k(n)$. Note that $S_k(n)$ can be evaluated for very large values of n using a computer algebra system. A discrepancy in this approach arises from the unknown function ϕ_k. Albeit asymptotically bounded, it can introduce large errors for finite values of n.

It was already noticed by Fröberg [5], that ϕ_k oscillates with a small (and decaying) amplitude around a constant value. We used our extensive data for $a_k(n)$ to look more closely at

$$\phi_k(n) \equiv \ln a_k(n) - \ln \sum_{j=0}^{\infty} \frac{n^j}{k^{\binom{j}{2}} j!} \,. \tag{18}$$

As can be seen from Figure 2, the amplitude of ϕ_2 is smaller than 10^{-4} for $n > 10^4$, and it is slowly, but monotonically

j	$\lvert\alpha_j(2)\rvert$	$\lvert\alpha_j(3)\rvert$
1	$7.36616 \cdot 10^{-7}$	$1.15010 \cdot 10^{-4}$
2	$4.63909 \cdot 10^{-13}$	$1.45894 \cdot 10^{-8}$
3	$2.64857 \cdot 10^{-19}$	$2.10798 \cdot 10^{-12}$
4	$1.29245 \cdot 10^{-25}$	$1.45000 \cdot 10^{-16}$

Table II: Taylor coefficients (8) for ϕ_k and ψ_k.

decaying. The constant around which ϕ_k oscillates will cancel in the ratio

$$\frac{a_k(kn)}{a_k(n)} = e^{\phi_k(kn)-\phi_k(n)}\,\frac{S_k(kn)}{S_k(n)}$$

Hence the error is bounded by the small amplitude. This is confirmed by the numerical data, see Table I. Even for $n = 100$, the error in $c_2(n)$ is only in the fourth decimal.

This observation tells us that we can use

$$b_k(n) = \frac{S_k(kn)}{S_k(n)}\,\frac{\ln n}{n} \tag{19}$$

as an excellent approximation to $c_k(n)$. Since $b_k(n)$ can be evaluated for very large values of n, like $n = 2^{1000}$ and beyond, we can use $b_k(n)$ to bridge the gap between the numerically accessible $c_k(n)$ and $c_k(\infty)$ (Figure 1).

IV. ASYMPTOTICS RELOADED

The results of de Bruijn (6) and Mahler (17) have to match, i.e., we know that $\phi_k(n) + \ln S_k(n)$ equals the right hand side of (6). A saddle-point expansion of $S_k(n)$ (see (36) in the Appendix) reveals that the leading terms of $\ln S_k(n)$ equal the leading terms in (6). The remaining terms yield

$$\phi_k(n) = \Psi_k\left(\log_k\left(\frac{n}{\log_k n}\right)\right) + \left(1+\frac{\ln\ln k}{\ln k}\right)\ln k - \frac{1}{2}\ln(2\pi) + \frac{(\ln k + 2\ln\ln k)^2}{8\ln k} + \mathcal{O}\left(\frac{\ln^2\ln n}{\ln n}\right). \tag{20}$$

In particular, we see that asymptotically ϕ_k oscillates around a value

$$\left(1+\frac{\ln\ln k}{\ln k}\right)\ln k - \frac{1}{2}\ln(2\pi) - \frac{(\ln k + 2\ln\ln k)^2}{8\ln k} + \alpha_0(k)\,, \tag{21}$$

with $\alpha_0(k)$ from (9). For $k = 2$, this constant is $-0.079793025\ldots$ (see Figure 2), in perfect agreement with the numerical results of Fröberg [5].

The asymptotic amplitude of ϕ_k is very small, as can be seen by evaluating the coefficients (8), see Table II. Hence we know that the oscillation in Figure 2 will eventually decay to an amplitude of size 10^{-6}. We have calculated a few more minima and maxima of ϕ_2 to check this decay. Figure 3 shows the result. The extrapolation of the numerical data gives very accurate result for the constant $-0.079793025\ldots$ as well as the right order of magnitude (10^{-6}) of the remanent amplitude.

V. ACKNOWLEDGMENTS

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VI. APPENDIX

Here, we evaluate the asymptotic behavior of the sum

$$S_k(n) = \sum_{j=0}^{\infty} \frac{n^j}{j!\,k^{\frac{j(j-1)}{2}}} \qquad (n \to \infty) \tag{22}$$

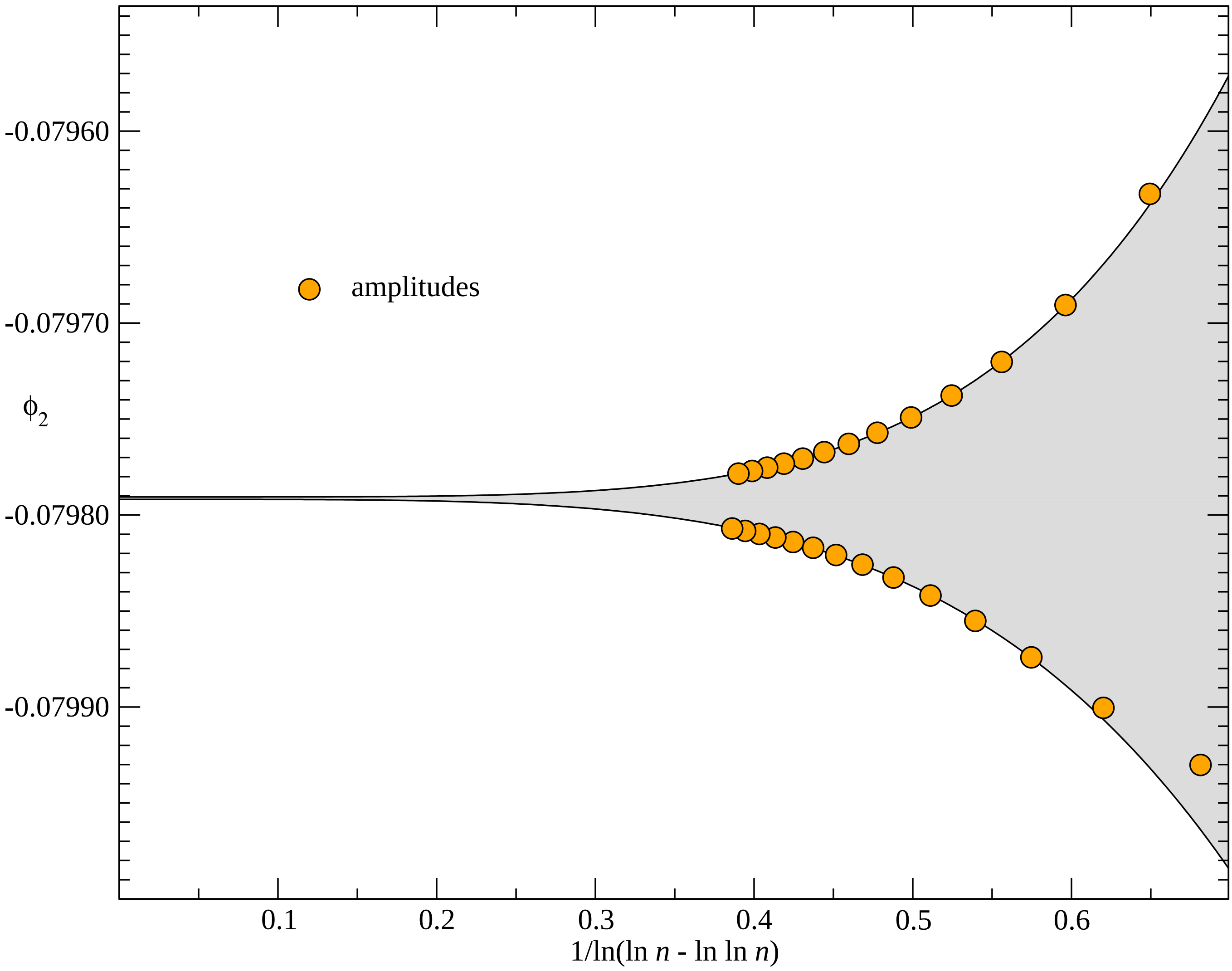

Figure 3: Decay of the amplitude of ϕ_2. The scale of the abscissa is chosen to match 1 over the period of ψ_k. The numerical fits for the minima and the maxima are three parameter least square fits of $\mu_0 + \mu_1 x^{\mu_2}$.

using a saddle-point expansion. Following Ref. [6] (pp. 304), we define $\Phi_j = \ln a_j$ for the summand a_j. The finite-difference condition $D\Phi_j = \Phi_j - \Phi_{j-1} = 0$ determines the maxima, i. e. we need to find the j_0-term(s) of the sum with $a_{j_0}/a_{j_0-1} \sim 1$. Applied to Eq. (22), we obtain

$$\eta\left[j_0 + \Delta_j - 1 + \log_k\left(j_0 + \Delta_J\right)\right] \;\sim\; 1 \tag{23}$$

where we use the abbreviations

$$\eta = \frac{1}{\log_k n} \;=\; \frac{\ln k}{\ln n} \tag{24}$$

and a non-integer offset $0 \le \Delta_j \le 1$ on the integer location j_0 of the saddlepoint, which we will need to attain the continuum limit for this n-dependent ("moving") saddle point [6]. For $n \to \infty$, $\eta \to 0$ and in that limit we find from Eq. (23) for the saddle-point location by peeling off layer-by-layer:

$$\begin{aligned} j_0 \;=\; & \frac{1}{\eta} + \log_k \eta + 1 - \Delta_j - \eta \frac{1 + \log_k \eta}{\ln k} + \eta^2 \frac{1 + \log_k \eta}{2 \ln k}\left[\frac{2}{\ln k} + 1 + \log_k \eta\right] \\ & - \eta^3 \frac{1 + \log_k \eta}{6 \ln k}\left[\frac{6}{\ln^2 k} + \frac{9}{\ln k} + 2 + \frac{\ln \eta}{\ln k}\left(4 + \frac{9}{\ln k}\right) + 2\frac{\ln^2 \eta}{\ln^2 k}\right] + O\left(\eta^4 \ln^4 \eta\right). \end{aligned} \tag{25}$$

As $j_0 \gg 1$ for $\eta \to 0$, we can expand Φ_j for large arguments:

$$\begin{aligned} \Phi_j &= j\ln(n) - \ln(j!) - \frac{j(j-1)}{2}\ln(k), \\ &= \frac{j}{\eta}\ln(k) - \frac{j(j-1)}{2}\ln(k) + \left(j+\frac{1}{2}\right)\ln(j) + j - \frac{1}{2}\ln(2\pi) + \frac{1}{12j} + O\left(\frac{1}{j^3}\right), \end{aligned} \tag{26}$$

where we have used the Stirling expansion for the factorial to the necessary order.

At the (unique) maximum of Φ_j we set $j \sim j_0 + t$ and expand here only to quadratic (Gaussian) order in t:[7]

$$\begin{aligned} \Phi_{j_0+t} &= \frac{j_0}{\eta}\ln(k) - \frac{j_0(j_0-1)}{2}\ln(k) + \left(j_0+\frac{1}{2}\right)\ln(j_0) + j_0 - \frac{1}{2}\ln(2\pi) + \frac{1}{12j_0} \\ &\quad + t\left[\left(\frac{1}{\eta} - j_0 + \frac{1}{2}\right)\ln(k) - \frac{\ln(j_0)}{2} - \frac{1}{2j_0} + \frac{1}{12j_0^2}\right] - t^2\left[\frac{\ln(k)}{2} + \frac{1}{2j_0} - \frac{1}{4j_0^2}\right] + O\left(\frac{t^3}{j_0^2}\right). \end{aligned} \tag{27}$$

Note that the linear term in t only vanishes (indicating a symmetric maximum) after we insert the moving saddle-point in Eq. (25) and Δ_j is fixed:

$$\begin{aligned} \Phi_{j_0+t} &= \frac{\ln(k)}{2\eta^2} + \frac{2\ln(\eta)+\ln(k)+2}{2\eta} + \left[\frac{\ln^2(\eta)}{2\ln(k)} + \ln(\eta) - \frac{1}{2}\ln(2\pi) + \frac{\Delta_j - \Delta_j^2}{2}\ln(k)\right] \\ &\quad -\eta\left[\frac{\ln^2(\eta)}{2\ln^2(k)} + \frac{\ln(\eta)}{\ln(k)} + \frac{7}{12} - \frac{\Delta_j-\Delta_j^2}{2}\right] \\ &\quad +\frac{\eta^2}{12}\left[\left(1+\frac{\ln\eta}{\ln k}\right)\left(6\frac{1+\frac{\ln\eta}{\ln k}}{\ln k} + 3 + 4\frac{\ln\eta}{\ln k} + 2\frac{\ln^2\eta}{\ln^2 k}\right) + \left(\Delta_j - \Delta_j^2\right)\left(2\Delta_j - 7 - 6\frac{\ln\eta}{\ln k}\right)\right] \end{aligned} \tag{28}$$

$$+O\left(\eta^3\ln^4\eta\right) \tag{29}$$

$$+\left[\frac{1}{2}\left(\ln(k)+\eta\right)\left(1-2\Delta_j\right) + \eta^2\left(\frac{7}{12} - \frac{3}{2}\Delta_j + \frac{1}{2}\Delta_j^2 + \frac{1-2\Delta_j}{2}\frac{\ln\eta}{\ln k}\right)\right]t \tag{30}$$

$$-\frac{1}{2}\left[\left(\ln(k)+\eta\right)\left(1-2\Delta_j\right) + \eta^2\left(\Delta_j - \frac{3}{2} - \frac{\ln\eta}{\ln k}\right)\right]t^2 + O\left(\eta^2 t^3\right). \tag{31}$$

Terms of orders such as $\eta^2 t^3$ will not contribute at order η^2 as they are at leading order asymmetric in t in the ensuing Gaussian integration. To that effect, we symmetrize the saddle point to order η^2 with the choice of

$$\Delta_j = \frac{1}{2} + \frac{\eta^2}{24\ln k} + O\left(\eta^3\ln\eta\right), \tag{32}$$

which also impacts constant or smaller terms in η in Eq. (30). The Gaussian integration then yields

$$S_k(n) \sim \sum_{t=-\epsilon j_0}^{\epsilon j_0} e^{\Phi_{j_0+t}}, \qquad (\eta \ll \epsilon \ll 1) \tag{33}$$

$$\begin{aligned} &= e^{\Phi_{j_0}}\int_{-\infty}^{\infty} dt\, \exp\left\{-\frac{1}{2}\left[\ln(k)+\eta-\eta^2\left(1+\frac{\ln\eta}{\ln k}\right)\right]t^2 + O\left(\eta^2t^3\right)\right\}, \\ &= e^{\Phi_{j_0}}\int_{-\infty}^{\infty} dt\, \left[1+O\left(\eta^2t^3\right)\right]\exp\left\{-\frac{1}{2}\left[\ln(k)+\eta-\eta^2\left(1+\frac{\ln\eta}{\ln k}\right)\right]t^2\right\}, \\ &= \exp\left\{\Phi_{j_0} + \frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\left[\ln(k)+\eta-\eta^2\left(1+\frac{\ln\eta}{\ln k}\right)\right] + O\left(\eta^3\ln^2\eta\right)\right\}. \end{aligned} \tag{34}$$

Note that the $\ln(2\pi)$-terms cancel. Hence, we finally obtain

$$\begin{aligned} \ln\left[S_k(n)\right] &= \frac{\ln k}{2\eta^2} + \frac{2\ln\eta+\ln k+2}{2\eta} + \left[\frac{\ln^2\eta}{2\ln k} + \ln\eta - \frac{1}{2}\ln\ln k + \frac{1}{8}\ln k\right] \\ &\quad -\eta\left[\frac{\ln^2\eta}{2\ln^2 k} + \frac{\ln\eta}{\ln k} + \frac{11}{24} + \frac{1}{2\ln k}\right] \\ &\quad +\eta^2\left[\frac{\ln^3\eta}{6\ln^3 k} + \left(1+\frac{1}{\ln k}\right)\frac{\ln^2\eta}{2\ln^2 k} + \left(\frac{11}{24} + \frac{3}{2\ln k}\right)\frac{\ln\eta}{\ln k} + \left(\frac{1}{8} + \frac{1}{\ln k} + \frac{1}{4\ln^2 k}\right)\right] \\ &\quad +O\left(\eta^3\ln^4\eta\right), \end{aligned} \tag{35}$$

or in terms of powers of $\ln n$ directly:

$$\begin{aligned}\ln\left[S_k\left(n\right)\right] &= \frac{\ln^2 n}{2\ln k}+\ln n\left[-\frac{\ln\ln n}{\ln k}+\frac{\ln\ln k}{\ln k}+\frac{1}{2}+\frac{1}{\ln k}\right]+\frac{\ln^2\ln n}{2\ln k}-\ln\ln n\left[\frac{\ln\ln k}{\ln k}+1\right]+\frac{\left[\ln k+2\ln\ln k\right]^2}{8\ln k}\\ &\quad -\frac{\ln k}{\ln n}\left[\frac{\ln^2\ln n}{2\ln^2 k}-\frac{\ln\ln n}{\ln k}\left(1+\frac{\ln\ln k}{\ln k}\right)+\frac{11}{24}+\frac{1}{2\ln k}+\frac{\ln\ln k}{\ln k}+\frac{\ln^2\ln k}{2\ln^2 k}\right]\\ &\quad -\frac{\ln^2 k}{\ln^2 n}\left[\frac{\ln^3\ln n}{6\ln^3 k}-\frac{\ln^2\ln n}{2\ln^2 k}\left(1+\frac{1}{\ln k}+\frac{\ln\ln k}{\ln k}\right)\right.\\ &\quad +\frac{\ln\ln n}{\ln k}\left(\frac{11}{24}+\frac{3}{2\ln k}+\frac{\ln\ln k}{\ln k}+\frac{\ln\ln k}{\ln^2 k}+\frac{\ln^2\ln k}{2\ln^2 k}\right)\\ &\quad \left.-\frac{1}{8}-\frac{1}{\ln k}-\frac{1}{4\ln^2 k}-\frac{11\ln\ln k}{24\ln k}-\frac{3\ln\ln k}{2\ln^2 k}-\frac{\ln^2\ln k}{2\ln^2 k}-\frac{\ln^2\ln k}{2\ln^3 k}-\frac{\ln^3\ln k}{6\ln^3 k}\right]\\ &\quad +O\left(\frac{\ln^4\ln n}{\ln^3 n}\right).\end{aligned}\tag{36}$$

In Fig. 4 we plot a sequence of approximants to the numerically exact evaluation of the sum in Eq. (22), which prove to approximate with an error of the indicated order.

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(Concerned with sequence A033485, A000123, A005704, A005705, and A005706.)

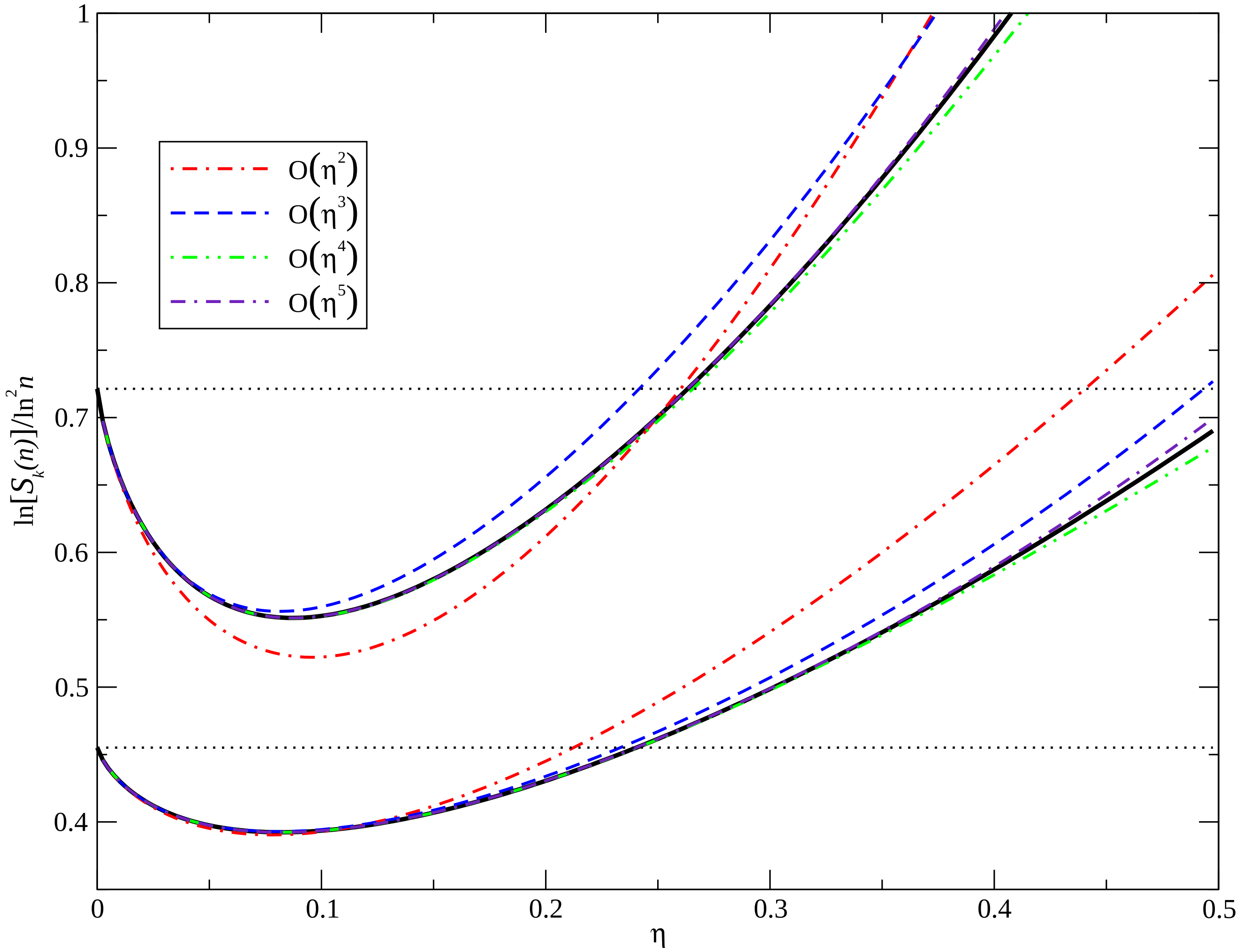

Figure 4: Plot of $\ln S_k(n)/\ln^2 n$ vs $\eta = \ln k/\ln n$ for $k = 2$ (top set) and 3 (bottom set). The dotted horizontal lines specify the asymptotic limits, $\frac{1}{2\ln 2} = 0.7213\ldots$ for $k = 2$ and $\frac{1}{2\ln 3} = 0.4551\ldots$ for $k = 3$. The thick black line is obtained from the numerically exact evaluation of Eq. (22), and the shaded, dashed lines correspond to the asymptotic expression in Eq. (35), evaluated to the indicated order in η.