Doppler effect – kinematics versus dynamics, conservation laws, and a small detour about length contraction

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There has to be a follow-up to my last essay on the Doppler effect, where I stated that I don't know of any derivation of the velocity Doppler effect based on energy conservation. Stefano Quattrini pointed out a few papers to me where energy conservation is indeed used in derivations of variants of the Doppler effect.

However, the way it is used does not suggest the Doppler effect to be a consequence of energy conservation, so I would still be reluctant to say that these derivations are *based on* energy conservation. The Doppler "effect" essentially is the transformation law for frequencies between different frames of reference. It is not a physical process. If the frequency of the waves under consideration can be related to an energy, which certainly is the case for photons or phonons, then you may be able to derive the transformation law for frequencies from the known transformation laws for energy and momentum of non-wavy objects, if you can express the energy and momentum connected with the wave properties (i.e., frequency, wave vector) by the energies of those objects involved in an interaction process with the wave. The conservation laws hold in either frame. The energy-momentum of the photon, say, can then be expressed by differences of energy-momenta of other objects and knowing how to transform these, you will get the transformation laws for the frequency and wave vector, i.e., the Doppler effect and aberration.

Hence, the application of the conservation laws to some process (which itself is *not* the Doppler effect) in two frames may allow you to derive the Doppler effect. Since there are other methods to obtain it, I would consider this an *auxiliary* use of the conservation laws, rather than a fundamental one. Just that the frequencies involved in the Doppler effect appear in a law of energy conservation should not be taken as indication that the effect follows from energy conservation. What it shows at best is that the effect is not incompatible with energy conservation, and this is of course to be expected.

This, I believe, should at least be maintained for the basic Doppler effect, describing a frequency change between different frames of reference. In this essay, I will however mostly consider the *double Doppler effect*, a notion I take from Ref. [1].[1] This concept is useful in the description of Doppler radar velocity measurements, where an electromagnetic wave is sent to an object from which it is reflected back to the emitter. The frequency change between the emitted and reflected waves is the ratio of two frequencies in a *single* frame of reference (that of the emitter), so its relationship to energy conservation is certainly worth discussion.

Before describing and analysing the experiment as well as deriving an exact double Doppler formula for the case, where recoil effects on the reflecting object are taken into account, I would like to make a small detour highlighting the relationship between kinematics and dynamics for another phenomenon, which is not a process either, *viz.* the relativistic length contraction. There are analogies between the way kinematics and dynamics get intertwined in discussions of length contraction and those of the Doppler effect.

[1] My own name for it was two-way Doppler effect. The basic Doppler effect would then be the one-way Doppler effect. One analogy with the two-way and one-way speeds of light is that the two-way effect can be calculated easily from the one-way case, but not vice versa. Knowledge of the two-way speed of light does not allow you to calculate the one-way speed of light without further information, in this case on the synchrony. Knowledge of the frequency shift of the two-way Doppler effect does not allow you to derive the one-way effect without additional information. In the electromagnetic case in vacuum, such a piece of information would be the relativity principle, i.e., a statement about the relationship between physical laws in different frames of reference.

Length contraction

In his paper introducing special relativity [2], Einstein discusses the "relativity of lengths and times", including the phenomenon that a length will be measured[2] to be shorter in an inertial frame in which it is moving than in its (momentary inertial) rest frame, but he never uses the name of *length contraction* that has caught on since. Actually, this is a misnomer, because no process of contraction is involved. Rather, it means that the length of an object in a given state of motion is measured to be different in a frame where it is at rest than in a frame where it is moving, and it is shorter in the latter frame than in the former, by a factor depending on the relative velocity of the two frames only. So it is the *same* object in *one* state having different lengths in *two* different frames, not *one* object having different lengths in the *same* frame in *two* different states of motion.[3]

Yet it is the latter situation that is often considered to correspond to the meaning of length contraction, and not necessarily for bad reasons. When a spaceship is accelerated from rest to a certain velocity and coasts without acceleration afterwards, then its length changes between take-off and the final constant-velocity state. It really contracts and this is a dynamical process, due to the fact that an acceleration program thrusting the craft ahead in a way that keeps mechanical stresses inside small (avoiding to destroy the spaceship) must accelerate the tip of the rocket less than the bottom. This way the length of the craft shrinks in the stationary frame (from which take-off took place), while all proper lengths aboard including the total length of the craft remain roughly constant (and take exactly their original values once the engine is switched off and external mechanical forces disappear). Clearly, this dynamical length contraction is described by the same formula as the kinematical one introduced by Einstein, provided the spaceship's rest length before acceleration was the same as its rest length is now, in the constant-velocity state. But this is a condition that must be satisfied.

If, on the other hand, the rest length of a system changes on acceleration, the dynamical interpretation of length contraction may fail, and this has given rise to a number of paradoxes, most notably Bell's spaceship paradox and Ehrenfest's paradox.

Bell popularized a scenario invented by Dewan and Beran [3], in which two spaceships accelerate at exactly the same instantaneous acceleration in some stationary frame, with a taut rope connecting them (along the direction of motion). The distance of the spaceships must remain constant, as they have the same velocity at any time. So the length of the rope must also remain constant, even though it "should length contract" (dynamically, that is). In fact, the rope *is* length contracted at any point of its journey while it is moving and still intact. Its – constant – length is smaller by the appropriate factor $1/\gamma$ of any momentarily comoving inertial frame, in which it is (approximately) at rest, meaning that its length *in that frame* is larger by $\approx \gamma(v)$ than the constant length between the two spaceships in the stationary frame. Since the velocity v increases during acceleration that means that the rope becomes more and more stretched.[4] The ensuing tensile forces will eventually make it break.

Ehrenfest's paradox exhibits an even more dramatic failure of the dynamic interpretation of length contraction. A circular disk of radius R has a circumference $2\pi R$. Now set it in rotation about its center so that the circumference moves at speed v. According to Ehrenfest, each length element of the circumference must then undergo length contraction (by a factor of

[2] He defines the length measurement of a moving object to be the measurement of a distance at a fixed time.

[3] Note the similarity with the basic Doppler effect, ascribing different frequencies to the same state of spatio-temporal oscillation, i.e. wave, in two different frames. It is not a change of frequency between two states of the wave in the same frame.

[4] This is consistent with the fact that in order to keep the proper length of an accelerating object constant, its front end must be accelerated more slowly than its back end. For our rope, both ends are equally accelerated, so its proper length must increase (by elasto-plastic deformation), until rupture occurs.

$1/\gamma(v)$), so the circumference becomes smaller than $2\pi R$. But a rotating disk is still circular and the Euclidean geometry of the inertial system in which the disk rotates[5] simply requires that the circumference is $2\pi R$. This is a contradiction: the circumference cannot both be smaller than $2\pi R$ and equal to $2\pi R$.

Ehrenfest's paradox disappears, as soon as we realize the kinematic nature of length contraction. The rotating circumference *is* length contracted in the (inertial) frame of the disk center, but of course each length element is shorter than its proper length in a momentarily comoving inertial frame – not than its length before the disk was set in rotation. If one can "stitch together" all these momentarily comoving local inertial frames to form a non-inertial frame, in which disk points are stationary, then the circumference of the disk in this frame must be larger than $2\pi R$ by a factor of $\gamma(v)$. Which means, of course, that the geometry for observers sitting on the disk is non-Euclidean, as the radius of the disk remains R but its circumference is $2\pi\gamma(v)R$.

The reason dynamic length contraction does not work here is that setting the disk in rotation uniformly, we have, for each length element of the circumference, a Bell's spaceship scenario. The two ends of the length elements are accelerated the same way, so the piece of material between them will get stretched the same way as the rope in the spaceship case. There will then be tensile stresses in the circumference of the rotating disk (and achievable velocities will be limited by the resistance of the disk material to stretching). As long as the disk supports these stresses, length elements of its circumference will increase their proper length via elastic or plastic deformation. Length contraction of course refers the length in the non-rotating frame to this elasto-plastically extended length in the corotating frame, not to the length before rotation. You can find a detailed discussion of both paradoxes in my paper [4].

The upshot of this little detour is that kinematic phenomena such as length contraction may be used to make predictions for dynamical effects (such as the length change on a change of velocity), if proper care is taken (the rest length of the object must not change between the initial and final dynamical states). While length contraction as defined by Einstein is not a dynamical process itself, it may become useful in the description of dynamical processes.

I think the same is true for the Doppler effect and will try to show this in the following. The example of length contraction may then be a useful reminder of how things work, when kinematical constraints are used to obtain dynamical results.

Reflection from a moving mirror

With the Doppler radar method (by which the police likes to catch drivers for speeding), a radar signal is sent towards a moving object (e.g., a car) and it is (partially) reflected and received by the emitter again. From the frequency shift of the returned signal, the velocity of the object is determined (and a ticket is issued if it exceeds the prescribed limit). Because there is a Doppler shift of the frequency of the signal in the frame of the receiver and a second one after reflection, now in the frame of the emitter, who is the second receiver, some authors call this double Doppler effect [1], and I will, too. For the sake of simplicity, I will consider only the case here (as I have done in my previous derivation of the one-way Doppler effect), where the velocity of the moving object and the signal are aligned (i.e., parallel or antiparallel). However, I will be more general than usual in *not* assuming a priori that the mass of the object is so large that its recoil under interaction with the radiation can be neglected. That is, my calculation covers the particular case of backscattering with the Compton effect as well (i.e.,

[5]Its mass is assumed not to be large enough to curve spacetime significantly.

the reflecting object could be as light as an electron). Later, I will make an argument using optical frequencies, which would correspond to lidar rather than radar. With electromagnetic waves in the visual range, we may imagine the reflector to simply be a mirror.

The problem can be treated at the single-photon level and this yields the full result already, if the reflector is massive enough so it does not change its velocity due to the interaction with small numbers of photons, because then the interaction between the reflector and the photons is unchanged from one photon to the next and the frequency change factor is the same for all photons. For a reflector having very small mass, its motion changes significantly by recoil, and subsequently arriving photons will suffer different frequency shifts.

My first approach focuses on a description of the process entirely in the frame of reference of the emitter. Fig. 1 visualizes the situation, which is essentially described as an elastic collision.

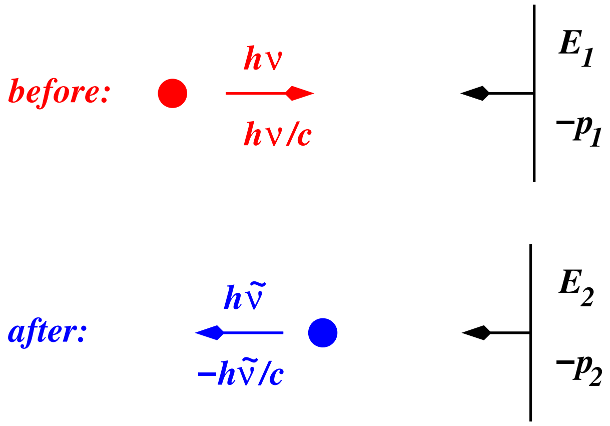

Fig. 1: Before the photon hits the mirror, it propagates to the right with energy $h\nu$ and momentum $h\nu/c$, while the mirror has the total momentum $-p_1$ and energy $E_1 = \sqrt{M^2c^4 + p_1^2c^2}$, where M is the mass of the mirror; depicted is the situation, where the mirror moves to the left (so the sign of the momentum is negative). After collision, the photon moves to the left, with energy $h\tilde{\nu}$, its momentum then is $-h\tilde{\nu}/c$. The mirror retains the momentum $-p_2$ and the energy $E_2 = \sqrt{M^2c^4 + p_2^2c^2}$.

Energy and momentum conservation for the collision provide the following two equations:

$$h\nu + \sqrt{M^2c^4 + p_1^2c^2} = h\tilde{\nu} + \sqrt{M^2c^4 + p_2^2c^2}\,, \tag{1}$$

$$\frac{h\nu}{c} - p_1 = -\frac{h\tilde{\nu}}{c} - p_2\,, \tag{2}$$

and it is useful to note that the velocity of the mirror can be directly calculated from its energy and momentum:[6]

$$v_i = \frac{p_i c^2}{E_i} \qquad i = 1, 2 \tag{3}$$

We rearrange (2) and (1) a bit:

$$\frac{h(\nu + \tilde{\nu})}{c} = p_1 - p_2\,, \tag{4}$$

$$\frac{h(\nu - \tilde{\nu})}{c} = \sqrt{M^2c^2 + p_2^2} - \sqrt{M^2c^2 + p_1^2} = \frac{p_2^2 - p_1^2}{\sqrt{M^2c^2 + p_2^2} + \sqrt{M^2c^2 + p_1^2}}$$

[6] We have $p_i = M\gamma(v_i)v_i = M\left(1 - (v_i^2/c^2)\right)^{-1/2} v_i \;\Rightarrow\; p_i^2\left(1 - (v_i^2/c^2)\right) = M^2v_i^2 \;\Rightarrow\; p_i^2 = \left(M^2 + p_i^2/c^2\right)v_i^2 \;\Rightarrow\; v_i^2 = p_i^2c^2/(M^2c^2 + p_i^2) = p_i^2c^4/E_i^2$

$$= -\frac{p_1^2 - p_2^2}{E_1/c + E_2/c} \,. \tag{5}$$

Taking the ratio of (5) and (4), we finally get

$$\begin{aligned}\frac{1-\tilde{\nu}/\nu}{1+\tilde{\nu}/\nu} &= -\frac{p_1+p_2}{(E_1+E_2)/c} = -\frac{p_1c^2+p_2c^2}{c(E_1+E_2)} = -\frac{1}{c}\left(\frac{E_1}{E_1+E_2}\frac{p_1c^2}{E_1} + \frac{E_2}{E_1+E_2}\frac{p_2c^2}{E_2}\right) \\ &= -\frac{1}{c}\left(\frac{E_1}{E_1+E_2}v_1 + \frac{E_2}{E_1+E_2}v_2\right) \equiv -\frac{v}{c}\,,\end{aligned} \tag{6}$$

where we have used (3) to introduce the velocities of the mirror before and after reflection. The final equality defines a weighted average of these velocities

$$v = \alpha v_1 + (1-\alpha)v_2, \qquad \alpha \equiv \frac{E_1}{E_1+E_2}\,, \tag{7}$$

with the weight factors given by the ratios between the energies of the mirror before/after reflection and the sum of these energies. The physical significance of this weighting will be discussed later. The difference between E_1 and E_2 is smaller than the energy of the infalling photon, so in practice it will be negligible for macroscopic mirrors. We then have $\alpha = 1/2$ and $v_1 = v_2 = v$. When we are dealing with gamma rays and the "mirror" is an object of atomic size, there may be observable differences between the three velocities. In any case, Eq. (6) is an *exact* result. We may solve it for $\tilde{\nu}/\nu$ and obtain

$$\boxed{\frac{\tilde{\nu}}{\nu} = \frac{1+v/c}{1-v/c}} \tag{8}$$

which is the familiar formula for the Doppler radar frequency shift. It involves no approximation,[7] whether the recoil effect on the mirror is negligible or not. In case it is appreciable, we just have to take an appropriate average of the – then different – mirror velocities before and after reflection to keep the formula unchanged (and to avoid having two velocities appear explicitly in it). Of course, it also means that the measurement of $\tilde{\nu}$ for given ν does not yield the precise velocity of the object, which changes during the interaction anyway, it gives us an average of the velocities before and after interaction. If the complete initial state of the system is known, i.e., if we know ν, p_1 and E_1 (or the mass M, from which we can calculate E_1 knowing p_1), then the complete final state, given by $\tilde{\nu}$, p_2 and E_2, is calculable. This is not the main interest of this article, so I will not present the calculation here.[8] Note also that the same result holds for a bunch of n coherent photons moving together, all that changes is that h in the calculation is replaced by nh everywhere, and this factor cancels out in the frequency formula.

Have we derived the Doppler effect in obtaining Eq. (8)? Well, let's have a look at its definition in Wikipedia: *The Doppler effect (also Doppler shift) is the change in the frequency of a wave in relation to an observer who is moving relative to the source of the wave.*[9] Since the only

[7] Within the general theoretical framework, in which we are working. It is a result for flat spacetime, i.e., we are assuming that there are no gravitational fields around that would cause spacetime curvature.

[8] One way to proceed is to divide Eq. (1) by c and add the result to Eq. (2) to eliminate $\tilde{\nu}$. Then isolate the square root expression containing p_2 on one side of the equation and take the square to obtain an equation that is *linear* in p_2, because the p_2^2 terms cancel. This gives an expression determining p_2 in terms of p_1, M, and ν, all known quantities from the initial state. Once p_2 is known exactly, it is trivial to obtain E_2 and then we may use Eqs. (7) and (8) to obtain $\tilde{\nu}$.

[9] A slightly more precise way of formulating this would be: *The Doppler effect (also Doppler shift) is the change in the frequency of a wave between the frames of reference of an observer and of the source of the wave, who are in relative motion with respect to each other.* This clarifies that by "frequency in relation to" is meant "frequency in the frame of", i.e., both observer and source measure the frequency to which the effect refers in their rest frames, respectively. Neither adopts a description in terms of a frame that is moving w.r.t. themselves.

frequency change that we derived via energy and momentum conservation is not a change in relation to an observer moving relative to the source – the observer and the source are at rest with respect to each other, being part of a single device or experimental arrangement, this does not look much like a derivation of a Doppler effect. Moreover, our derivation was not based on wave properties, it used a particle picture for the electromagnetic signal, which was considered to be a photon. This is not a severe objection, however, because we are familiar with the wave-particle duality from quantum mechanics and the de-Broglie relations.

Nevertheless, all we did was to deduce a frequency shift in a certain experiment. To relate it to the Doppler effect and to show that it indeed is interpretable in terms of two successive Doppler effects, it would be beneficial to have a derivation that makes use of a frame of reference of a *moving* observer in addition to the single observer *at rest* appearing in our discussion so far.

We might be inclined to think that this is only possible, if the recoil is negligible, because otherwise we will have two velocities of the mirror, the pre-collision velocity, defining a moving observer frame of reference, and the after-collision velocity, defining a differently moving emitter frame of reference. The whole process would then involve three frequency shifts, the first between the sender of the Doppler radar signal and the mirror before arrival of the signal, the second a frequency shift between the mirror frames before the signal absorption and after its reemission (which might not be a Doppler shift) and the third the shift between the reemission of the signal and its reception back at the original emitter.[10] Fortunately, it is not necessary to introduce three frames of reference. We will consider the reflection process, describable as absorption and reemission, not in either of the two mirror frames but rather in a frame moving at an average velocity, defined on the basis of a criterion maximizing the simplicity of the description. As it will turn out, this velocity is given by Eq. (7) in the frame of reference of the original sender of the Doppler radar signal.

The crucial idea is to take a frame, in which the absorption and reemission of the photon happen at the same frequency (see Fig. 2).

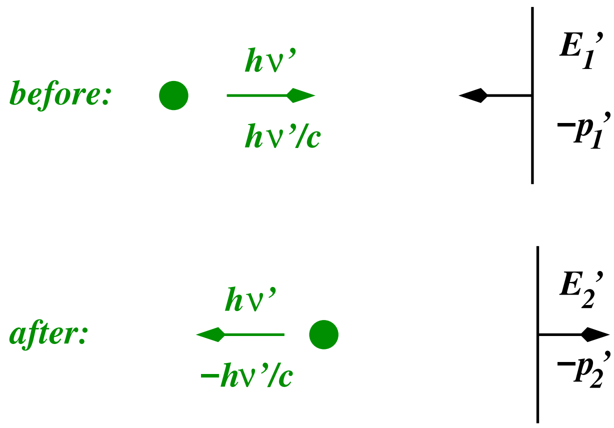

Fig. 2: Before the photon hits the mirror, it propagates to the right with energy $h\nu'$ and momentum $h\nu'/c$, while the mirror has the total momentum $-p_1'$ and energy $E_1' = \sqrt{M^2c^4 + {p_1'}^2c^2}$; depicted is the situation, where the mirror moves to the left (so the sign of the momentum is negative, that of p_1' positive). After collision, the photon moves to the left, with the *same* energy as before $h\nu'$, its momentum then is $-h\nu'/c$. The mirror momentum becomes $-p_2'$ and its energy $E_2' = \sqrt{M^2c^4 + {p_2'}^2c^2}$. We use $-p_2'$ for the momentum, although it will turn out that this momentum points to the right, as is correctly indicated by the arrow. With this sign convention, p_2' will become negative.

[10]Note added on 16 September 2024: In fact, there are four frequency shifts, as will be demonstrated in an addendum, two of them compensating each other.

Energy and momentum conservation now read

$$h\nu' + \sqrt{M^2c^4 + {p'_1}^2c^2} = h\nu' + \sqrt{M^2c^4 + {p'_2}^2c^2}\,, \tag{9}$$

$$\frac{h\nu'}{c} - p'_1 = -\frac{h\nu'}{c} - p'_2\,, \tag{10}$$

and the first equation obviously implies $|p'_1| = |p'_2|$, the second then becomes

$$2\frac{h\nu'}{c} = p'_1 - p'_2 = 2p'_1\,, \tag{11}$$

where $\nu' > 0$ makes $p'_2 = -p'_1$ necessary (i.e., p'_1 and p'_2 cannot have the same sign). Hence,

$$p'_1 = \frac{h\nu'}{c} = -p'_2\,, \tag{12}$$

and this is the entire result we get from energy and momentum conservation. The average frame of reference therefore is chosen so that the sum of the momenta before and after reflection becomes zero. (We could have taken that as the condition for the choice of the frame instead of equality of the frequencies before and after reflection. Then that equality would have been derived from energy and momentum conservation and the condition that "on average" the mirror is at rest.)

In order to relate the frequency ν' to the emission and absorption frequencies ν and $\tilde{\nu}$ in the original emitter's frame, energy and momentum conservation are of little use. However, since I derived the Doppler effect previously (in another essay, see `https://wasd.urz.uni-magdeburg.de/kassner/research_gate_pres/sci_edu_res_gate/derivation_doppler.html`), I simply can take over the results. All that is needed is to calculate the velocity of the emitter frame in the reflector frame. This is not difficult, using the transformation formulas for energy and momentum between the two frames. Let us call the sought-for velocity u. The Lorentz transformations between the two frames then read

$$\begin{aligned} ct &= \gamma\left(ct' - \frac{u}{c}x'\right)\,, \qquad \gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \\ x &= \gamma\left(x' - \frac{u}{c}ct'\right)\,, \end{aligned} \tag{13}$$

and the energy-momentum four vector (of which we consider only two components) must transform the same way:

$$\begin{aligned} \frac{E}{c} &= \gamma\left(\frac{E'}{c} - \frac{u}{c}p'\right)\,, \\ p &= \gamma\left(p' - \frac{u}{c}\frac{E'}{c}\right)\,. \end{aligned} \tag{14}$$

We simply write this out for the two momenta and energies $p = -p_1$, $E = E_1$ and $p = -p_2$, $E = E_2$:

$$-p_1 = \gamma\left(-p'_1 - \frac{u}{c^2}E'_1\right) = \gamma\left(-\frac{h\nu'}{c} - \frac{u}{c^2}\sqrt{M^2c^4 + h^2\nu'^2}\right)\,, \tag{15}$$

$$-p_2 = \gamma\left(-p'_2 - \frac{u}{c^2}E'_2\right) = \gamma\left(\frac{h\nu'}{c} - \frac{u}{c^2}\sqrt{M^2c^4 + h^2\nu'^2}\right)\,, \tag{16}$$

$$\Rightarrow \quad p_1 + p_2 = \frac{2\gamma u}{c^2}\sqrt{M^2c^4 + h^2\nu'^2}\,, \tag{17}$$

$$\frac{E_1}{c} = \gamma\left(\sqrt{M^2c^2 + \frac{h^2\nu'^2}{c^2}} + \frac{u}{c^2}h\nu'\right) , \tag{18}$$

$$\frac{E_2}{c} = \gamma\left(\sqrt{M^2c^2 + \frac{h^2\nu'^2}{c^2}} - \frac{u}{c^2}h\nu'\right) , \tag{19}$$

$$\Rightarrow \quad \frac{E_1 + E_2}{c} = 2\gamma\sqrt{M^2c^2 + \frac{h^2\nu'^2}{c^2}} \tag{20}$$

Taking now the ratio of (17) and (20), we get the second expression from Eq. (6):

$$\frac{p_1 + p_2}{(E_1 + E_2)/c} = \frac{u}{c^2}\frac{\sqrt{M^2c^4 + h^2\nu'^2}}{\sqrt{M^2c^2 + h^2\nu'^2/c^2}} = \frac{u}{c} \stackrel{!}{=} \frac{v}{c} , \tag{21}$$

demonstrating that the velocity u of the emitter defined here is precisely the negative of the velocity $-v$ of the average mirror frame defined in Eq. (7). That is, we may now justify the weight coefficients of this average by pointing out that it is exactly with this weighting by energies that the average mirror frame will be the rest frame of an observer for whom the photon is reflected without energy exchange (and where its momentum will just be reversed).

Hence, the emitter moves at velocity v in the average mirror frame, which serves as receiver for the incoming photon and as emitter for the reflected one. Applying the standard Doppler effect formula, we find then for the frequency ν, by which the photon was originally emitted:

$$\frac{\nu'}{\nu} = \sqrt{\frac{1 + v/c}{1 - v/c}} \qquad \Rightarrow \qquad \nu = \sqrt{\frac{1 - v/c}{1 + v/c}}\,\nu' . \tag{22}$$

Next, the original emitter becomes a receiver for the reflected photon, so the ratio between the final observed frequency $\tilde{\nu}$ and the frequency of the reflected photon is:

$$\frac{\tilde{\nu}}{\nu'} = \sqrt{\frac{1 + v/c}{1 - v/c}} . \tag{23}$$

Therefore, the total frequency shift is described by

$$\boxed{\frac{\tilde{\nu}}{\nu} = \frac{1 + v/c}{1 - v/c}} \tag{24}$$

which agrees with (8) but is now a result obtained using the Doppler shift between frames twice, thus justifying the name double Doppler effect.

It should however be kept in mind that by double Doppler effect we mean a dynamical process consisting of three subprocesses: emission of the electromagnetic wave and its propagation to the moving object, reflection (which is the process that here has been discussed in terms of energy-momentum conservation), and propagation back and reception of the wave. In the frame of the original emitter, the frequency shift happens during the reflection step, in the average frame associated with the reflector, no frequency shift happens at all. The reflector ascribes the fact that the original emitter notices a frequency shift to two ordinary Doppler shifts. These do not "happen" at the moments of initial emission or final reception. The frequency of the incoming photon is *always* different for the emitter than for the observer in the average reflector frame, and the frequency of the reflected photon is also different along all of its path for the two frames. This of course makes it impossible to interpret the Doppler shifts as being due to energy conservation. The process to which energy conservation was applied in both derivations, is the reflection process, but it leads to a frequency shift only in the first derivation (in the second, the frequency of the photon is unchanged on reflection), and that frequency shift is different from that of the Doppler effect.[11]

I would like to support, as I have done in my first essay on the Doppler effect, the argument against it being a consequence of energy conservation by a thought experiment, embellishing our Doppler radar setup. First, I would like to make it Doppler lidar instead of radar, i.e., I would like to work at optical frequencies. That is because at optical frequencies there exist color filters. Assume we have a set of color filters transparent to the frequency ν' but blocking the frequencies ν and $\tilde{\nu}$ (i.e., the filters will *not* absorb photons of the first frequency but absorb those of the two other frequencies).

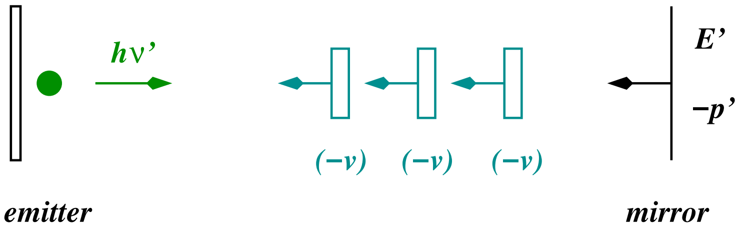

Fig. 3: Same setup as in Figs. 1 and 2, but with color filters added. These move at velocity $-v$ in the emitter frame (that is they are at rest in the average mirror frame, the mirror has momentum $-h\nu'/c$ in that frame before and momentum $h\nu'/c$ after reflection) and are transparent to frequency ν' but block frequencies ν and $\tilde{\nu}$.

Will the photon get through to the reflector and back to the emitter? Given the description in the average reflector frame, we can answer that question in the affirmative. In that frame, all the filters are at rest and the photon has the right frequency (ν') to pass in both directions. Note also that the observer in the average mirror frame can deduce from the fact that the photon successfully is sent to him and returns that it has the frequency ν' during all passages through a filter.

How is the emitter to explain that the photon passes all filters? After all, if the photon successfully gets emitted and returns according to the mirror observer, it cannot be blocked in his reckoning.[12] But the photon has a wrong frequency on both segments of its path, its frequency is too low (ν) on the way towards and too high ($\tilde{\nu}$) on the way back from the reflector. Well, that is true in the frame of the emitter...

But the emitter realizes that the filter "sees" the photon at at different frequency on the way to the mirror, because there is a Doppler effect, due to the motion of the filter at velocity $-v$ (which is towards the emitter). This Doppler effect will increase the frequency by a factor $(1+v/c)^{1/2}/(1-v/c)^{1/2}$, from ν to ν', which means it will just pass. It should be noted that this Doppler effect is not connected with any absorption or reflection, so it cannot be explained by an energy transfer to the photon. Indeed, the presence of this Doppler effect is inferred from the photon *not* being absorbed/blocked, i.e., from the fact that the filter is transparent to it, and it goes through without significantly interacting with the filter.

On the way back, the frequency of the photon is $\tilde{\nu}$ according to the emitter observer, and that is the case immediately after reflection. But now it approaches the filters from "behind", i.e., it catches up with them, so the Doppler effect reduces its frequency in the filter frame, by a factor $(1-v/c)^{1/2}/(1+v/c)^{1/2}$, and if you multiply $\tilde{\nu}$ by that, you again obtain ν'. This second Doppler effect also cannot be explained by some energy exchange with the filter,

[11]By which I mean the ordinary or basic Doppler effect, involving exactly one change of frames of reference.
[12]Emission of a signal and reception of a return signal are objective events. If they happen in one frame, they must happen in any frame.

because again there is no significant interaction between the photon and the filter, otherwise the photon would be absorbed. Note also that while the frequency is ν' for the filter, it remains $\tilde{\nu}$ for the emitter. There is no frequency shift in the emitter frame on either the way towards or the way back from the reflector, all of the frequency shift (from ν to $\tilde{\nu}$) happens during the reflection event.

I think this makes it pretty clear that the Doppler effect is kinematic and simply describes the frame dependence of the frequency of a wave; it is a phenomenon, not a process. The double Doppler effect, on the other hand, is a (sequence of) process(es), in which the kinematics of the Doppler effect plays a role, if a description using several frames of reference is used. The distinction between the two effects is similar to that between length contraction and dynamic length contraction, except that the latter seems more closely related to the kinematic phenomenon, because it is described by the same formula.[13] This is not the case with the double Doppler effect, which is quantitatively different from the Doppler effect. But still the dynamical phenomenon can be described quantitatively with the help of the kinematical one.

Double Doppler effect and the equivalence of inertial frames

Finally, I would like to comment on Quattrini's attempt at a sweeping attack on the relativity principle, where he uses a thought experiment involving an iterated double Doppler effect. His paper is deposited on a web page hosted by Research Gate:
`https://www.researchgate.net/publication/382239104_CONSERVATION_LAWS_-LIMITS_OF_THE_EQUIVALENCE_OF_INERTIAL_FRAMES`

Since he may change the paper in the future, I put a copy of the version that I am referring to (dated 13.07.2024) on the same web page as this article, where you can click on it.

He considers the radar Doppler experiment with a mirror moving towards the emitter of the waves and allows for repeated reflection, i.e., the emitter becomes a reflector after having fed the electromagnetic field into the system. The frequency of each photon increases by the double Doppler factor on each reflection at the moving mirror.[14] From this Quattrini concludes that in the course of time the energy density of the electromagnetic field will increase, which is correct under the idealizing assumptions of the thought experiment.[15] But then he concludes, and this is an erratic conclusion, that energy seems to be created out of nothing, a perpetuum mobile would arise, and energy conservation be violated, unless the formula for the double Doppler effect were modified. There are even weirder conclusions. I will discuss them afterwards.

The reason why a conclusion about the experiment being "at variance with conservation laws" is rash, to say the least, is that he considers only the energy of a component of the system, viz.

[13] If we include elastic and plastic deformations in the definition of the dynamic length contraction, then it will be described by a general formula that is different from that for the kinematic effect (and it may even become length expansion). An example is Gron's generalization of Hooke's law [5].

[14] There is no frequency increase in the (average) frame of the emitter for reflections by the emitter, so the double Doppler factor applies once for each bounce from the mirror. In the average frame of the mirror, the double Doppler factor applies for each bounce from the emitter, whereas there is no frequency change on reflection from the mirror.

[15] One such assumption is that the number of photons is constant. However, the quantum mechanical state of the electromagnetic field, if it starts out at radar frequencies need not even *have* a fixed photon number. In quantum field theory, a coherent field is described by a state for which the photon number is undefined. But even if the initial state has a fixed photon number, that number changes easily, for thermodynamic reasons – the chemical potential of a photon is zero, which means it can be freely created or destroyed to accommodate temperature changes. In a real arrangement, many thermal photons will be created as the field energy increases and those will be absorbed by the emitter and reflector material and carry energy away. Still, we may assume a number of idealizations in a thought experiment that in practice cannot be realized.

the electromagnetic field. But that energy can easily change, if there is energy exchange with other parts of the system. Energy conservation holds for a closed system, hence it may hold for the whole apparatus, if that is well isolated with respect to the environment, but of course the electromagnetic field can exchange energy with the mirror or the emitter and if the energy of the field increases, then the (kinetic) energy of the mirror and/or the emitter decreases in a way to keep the total energy constant.

It is easy to give a qualitative discussion of how the idealized system would behave in the course of time. First, to make it closed, we would have to avoid fixing the emitter position in the external world. Let us assume that both the emitter and the mirror can slide frictionless. An inertial system, in which the whole process could be described for an indefinite time, would be given by the center-of-energy system. In this system, both the emitter and mirror will initially move towards the center of energy, thus compressing the electromagnetic field between them. Each bounce of a photon on either the emitter or mirror will impart momentum on them, pushing them outward. This will slow down the motion of the piece hit by the photon, decreasing its kinetic energy and increasing the frequency of the photon. The radiation pressure will increase, decelerating both the emitter and the mirror further. This continues for a while until their relative velocity is zero.[16] Then they will be accelerated outward. From that point on, the energy density of the radiation field between the two reflectors will decrease, as the double Doppler effect for receding objects leads to a frequency reduction rather than an increase. This decreases the radiation pressure, diminishing the outward acceleration. Asymptotically, the system will approach a state in which the radiation field has frequency zero (i.e., it will tend to disappear) and the emitter and mirror will move apart at constant finite velocities, with kinetic energies that add up to the sum of the initial kinetic energies of both objects (in the center-of-energy frame) plus the energy of the initial radiation field (in the same frame). The time to reach that asymptotic state will be infinite, because at any finite time, the energy density of the radiation field will still be positive.[17]

To summarize, what Quattrini says about energy conservation in the system is largely incorrect. In the course of his exposition, he also asserts that the result for the double Doppler effect (which I derived here) is only an approximation, due to the velocity change of the mirror, as long as the latter does not have infinite mass, which in reality never is the case. My derivation shows clearly that this is wrong, at least for any single pair of bounces of the photon between the emitter and the mirror. The result (8) is exact, whether the mirror has a finite mass or an infinite mass. It is only the identification of v with the mirror velocity that requires a large mirror mass. Without that identification, the formula remains exact, with an appropriate average of the two mirror velocities before and after interaction with the photon determining the frequency shift. Clearly, the change of velocity between different bounces of photons may lead to the necessity of using the formula with different velocities. That means, the Doppler factor varies with time in Quattrini's thought experiment. But it does not mean that the fundamental formula is inexact.

In practice, the recoil of mirrors does not produce any significant effect in experiments with radar or optical frequencies. However, when gamma rays are used and the "mirror" is a

[16] Calculations for the velocity change on each bounce could proceed in a very similar way to the calculations done here in the emitter frame. The center of energy could be considered the emitter for both bounces on the mirror and the original emitter, and energy-momentum conservation be applied for bounces on either side. The average mirror or average emitter frame moving at a velocity between the velocities before and after the reflection events, could be taken as momentary inertial frame, if desired. That is however not even necessary, as the entire description may proceed in a single frame, if we use the first approach discussed above for the calculation of the energy shift.

[17] This presupposes that the reflectivity of the emitter and the mirror remains one at all frequencies, i.e., there are never absorption losses. An idealized experiment...

single atom, then recoil becomes important.[18] Because the atoms have a thermal velocity distribution, the Doppler shift does not have a fixed frequency but a frequency distribution, i.e., a finite linewidth. Recoil effects can increase the observed limewidth well above the natural one. Gamma ray transitions between energy levels of heavy atoms may have a very small natural linewidth, in principle allowing for spectroscopy experiments with high accuracy. But that accuracy could not be reached for a long time in real life, because the emission and absorption are accompanied by a large recoil of the atom. Then came Mößbauer. He found out that if you do gamma ray emission and absorption at low temperature with solids, then you may achieve recoilless transitions. At sufficiently low temperatures, the likelihood of momentum transfer to phonons decreases (because their number becomes small). So the probability for a gamma transition without phonons becomes nonzero (and attains a useful magnitude) and in that case the recoil is to the whole crystal lattice rather than to a single excitation (phonon). Because the mass of the crystal is large, recoil becomes negligible, leading to relative precisions of 10^{-15} and better for gamma ray frequency measurement (using the Doppler shift to achieve resonance between two gamma ray transitions). By virtue of the Mößbauer effect, it became possible for Pound and Rebka in 1959 to measure the gravitational redshift due to a height difference of about 20 m in the gravitational field of the earth.[19]

Returning to Quattrini's paper, there are even worse claims, as the title of the paper suggests. Indeed, somehow Quattrini seems to believe that the fact that the mirror will not move at constant velocity implies that the equivalence of inertial frames (i.e., the relativity principle) gets lost.[20] But why on earth should the fact that a frame of reference fixed to the mirror will not remain inertial – because the mirror performs an accelerated motion – allow one to draw *any* conclusions about properties of inertial frames? That idea is a category mistake. A frame of reference is not something that has to be attached to an object. When we wish to describe the gravitational two-body problem for a planet and its moon, then we do not attach

[18] An atom is not planar, so one-dimensional considerations as those made here, are not sufficient. Non-coaligned directions of momenta also play a role in momentum transfer.

[19] To detect the redshifted frequency which no longer was in resonance with the – nominally equal – gamma ray frequency of the detector, the detector was moved, so the Doppler effect could be used to tune the frequencies to become equal. The velocity necessary for this tuning then gave the frequency shift.

[20] Note added on 16 September 2024: Quattrini pointed out to me that this was not what he said. For him, inertial systems are equivalent by definition; what he claims is that this equivalence is inapplicable in his thought experiment. This is a somewhat strange level of differentiation from someone who normally is not as accurate in his wording. Regarding "invalidation for application", to be distinguished from "validity by definition", the first thing to be pointed out is that the principle we are talking about is the relativity principle (the notion "equivalence principle" is usually reserved for another concept that Einstein used in his work on general relativity) which is a *physical* principle that has been tested experimentally millions of times. You cannot test a definition experimentally. (That is why it is, for example, nonsense to consider Newton's second axiom a definition of force.) So the equivalence of inertial frames, i.e., the relativity principle, is not a definition, it is a statement about physics. Second, inertial systems are immaterial mental constructions. They can be "realized" approximately and only in finite regions of space, by use of clocks and distance measuring devices, but their existence as analytical tools does not depend on such a realization any more as the existence of, say, the number π relies on the existence of ideal circles in the real world. If inertial systems can be defined that *means* they are applicable, because their definition implies that they may serve as frame of reference, that is they can be used to be referred to. In flat spacetime, they can always be defined to be extended through all of it. In curved spacetime, their existence is necessarily restricted to a local part of the spacetime, so if you wish to work with inertial frames globally, you have to consider a whole set of them, not just a single frame. Fortunately, we know how to work with non-inertial frames which often can be used globally. Now, in the experiment at hand, we could define (and hence use or *apply*) an inertial system that is attached to the mirror before it is hit by a photon, i.e. it would be the rest system of the mirror. Once the mirror is hit by a photon, it will be accelerated for a short time, i.e. become non-inertial during that interval. But that does not mean that our once-defined inertial system does not exist anymore or becomes inapplicable. All that happens is that the inertial system continues to move at the pre-collision velocity of the mirror while the mirror changes its velocity. So the inertial system, which still can be used for description is no longer identical with the rest system of the mirror, rather the mirror moves in it. Or, to put it bluntly: inertial systems, being immaterial, do not experience recoil!

the frame of reference to either body. Rather we take as origin of our frame the center of mass, which may lie outside of both bodies. In my discussion of the double Doppler effect, I explicitly avoided to attach a frame of reference to the mirror, because that would render the description of recoil effects more difficult. But even if a frame of reference is fixed (for a sufficiently short time interval) to an object that does not move inertially, it can be a momentary inertial frame of reference. Quattrini doubts the equivalence of inertial frames, because in his thought experiment the mirror frame turns non-inertial.[21] However, there is no logical path from the fact that a frame fixed to a body may become non-inertial, to the conclusion that different inertial frames are not equivalent (in the sense of the relativity principle). Otherwise, the mere fact that there are bodies moving in an accelerated fashion, i.e. non-inertially, would lead to said non-equivalence, an idea that nobody so far has advanced. Again, by what logic does the existence of non-inertial motion affect the properties of inertial frames?

Another unfounded claim by Quattrini is that recoil is the "real reason" of the Doppler effect. In fact, the most precise measurements (in terms of relative precision) are those where recoil is avoided via quantum mechanics as in the Mößbauer effect.

In the conclusions of Quattrini's paper, he gives a summary, where essentially nothing except the first sentence (which is uncontroversial but clumsily formulated) is correct. In particular, his statement "considering a very superficial analysis, Doppler effect might look like an observer dependent problem" is off the mark. A *superficial* analysis of the *double Doppler effect* might lead to the conclusion that the Doppler effect has nothing to do with frame dependence (e.g., if you consider only the first derivation I gave). A *thorough* analysis, however, shows that frame dependence is the essence of the *Doppler effect* and that the superficiality *missing* this arises through consideration of a composite phenomenon only (the double Doppler effect), instead of carrying the analysis further to study the elementary phenomenon (which I have done before). I recommend to Quattrini to study the arrangement of Fig. 3 carefully, in which the Doppler effect arises at each of the color filters, without any net absorption or recoil taking place.

Addendum of 17 September 2024

Apparently, the main message of the preceding text has been largely missed, at least by Quattrini. It is that the frequency shift observed by the emitter is interpretable as (double) Doppler effect within a description employing at least two frames of reference, but that in the one-frame description presented first, it is no Doppler effect at all. Rather, it is entirely due to the photon-mirror collision. This oversight may be due to the fact that the two frames I have exclusively looked at are both very special, leading to a 100% separation of effects, so to speak, in the two descriptions. In the first, we have no Doppler effect, in the second, the whole frequency shift is only Doppler effects, whereas the collision does not induce any frequency shift. Therefore, I would like to give here the description of the effect in a general inertial frame, which will demonstrate that the "double Doppler shift" generally is due partly to the (primitive) Doppler effect (arising twice) and partly to a frequency shift due to the collision. So there is no question of energy and momentum conservation being responsible for the Doppler part of the effect. It is only responsible for the collision part of the shift. These are two separate things.

Another reason for writing this addendum is that Quattrini has produced an alternative formula for the double Doppler shift, containing an energy (though not the recoil energy) and allegedly proving that my formula is only an approximation. Of course, this is not the way to prove a formula to not be exact. You have to show, where an approximation is made in the

[21] So does the emitter frame, as I have discussed above.

formula under discussion, not to give a different formula that may just express the same thing in terms of different quantities. I set out to check Quattrini's alternative formula, expecting it to be correct but equivalent to mine. In an earlier one of his derivations, I found an error that made continuation of the calculation pointless. In the third version of his paper (sporting a slightly changed title) that I have also linked on my web pages (so it can be looked up by readers in case it is not the last one), he gives a derivation that contains three errors and one approximation. The first two errors *compensate* each other, thus resulting in a correct (and interesting) formula for the double Doppler effect (that I will show to involve *three* inertial reference frames). It turns out to be mathematically equivalent to my formula, which I will also demonstrate. The third error in Quattrini's calculation is also compensated, but by his approximation, so his end result involving the energy of the mirror is in fact exact – and also equivalent to my formula. A bit of irony comes from the observation that originally Quattrini claimed this formula to be exact and mine an approximation, whereas in the paper version three, he considers his formula to be an approximation while still not having demonstrated that mine is not exact, so presumably the exact formula is mine and the approximation is his. In reality, they are both exact. They simply express the same relationship via different variables. Here, I will derive the two formulas avoiding the errors of Quattrini's calculation.

The Doppler radar experiment as described in an arbitrary inertial frame

Let ν_o be the frequency of our photon, which was originally emitted at frequency ν by the emitter E, in the frame of some observer O who is moving at velocity w to the right (see Fig. 4)[22] and call the frequency of the returning photon in O's frame $\tilde{\nu}_o$.

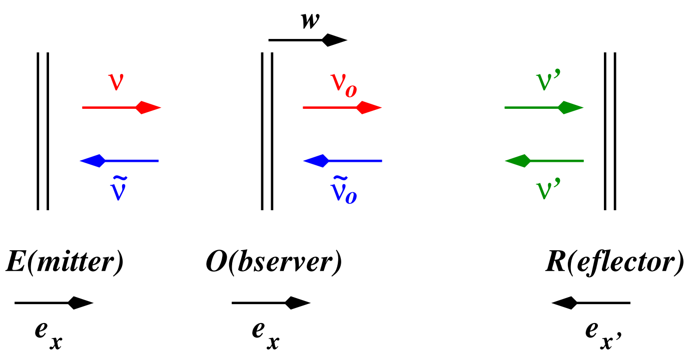

Fig. 4: An observer O moving at velocity w to the right would measure the incoming photon at frequency ν_o, the returning photon at frequency $\tilde{\nu}_o$. The collision is described in O's frame. Of course, the photon should not actually be absorbed by O (it could be measured in different experiments with the same setup – or its frequencies could simply be inferred from those measured by E).

Obviously, we can relate ν_o to ν and $\tilde{\nu}_o$ to $\tilde{\nu}$ by the standard Doppler effect formula:

$$\nu_o = \sqrt{\frac{1-w/c}{1+w/c}}\,\nu\,, \qquad \tilde{\nu} = \sqrt{\frac{1-w/c}{1+w/c}}\,\tilde{\nu}_o\,. \tag{25}$$

Next, we write down energy and momentum conservation for the collision in this frame:

$$h\nu_o + \sqrt{M^2c^4 + {p_1^{(o)}}^2 c^2} = h\tilde{\nu}_o + \sqrt{M^2c^4 + {p_2^{(o)}}^2 c^2}\,, \tag{26}$$

[22] It is not necessary that there really be an observer. All we require is the existence of an inertial frame moving at velocity $w\boldsymbol{e}_x$ with respect to the emitter frame. Its existence is guaranteed (via the relativity principle) by the existence of an inertial frame in which the emitter is at rest. The case of an emitter that changes velocity as well, due to recoil, will be briefly addressed later.

$$\frac{h\nu_o}{c} - p_1^{(o)} = -\frac{h\tilde{\nu}_o}{c} - p_2^{(o)} \,, \tag{27}$$

where the notation should be obvious: the momenta, in $O's$ frame, of the mirror before and after collision are characterized by a superscript (o). We then express, in the same way as with equations (4) and (5), the sum $h(\nu_o + \tilde{\nu}_o)$ via the second, the difference $h(\nu_o - \tilde{\nu}_o)$ via the first equation and take the ratio to obtain

$$\begin{aligned}
\frac{\nu_o - \tilde{\nu}_o}{\nu_o + \tilde{\nu}_o} &= \frac{\sqrt{M^2c^4 + {p_2^{(o)}}^2 c^2} - \sqrt{M^2c^4 + {p_1^{(o)}}^2 c^2}}{\left(p_1^{(o)} - p_2^{(o)}\right) c} \\
&= -\frac{\left(p_1^{(o)} + p_2^{(o)}\right) c}{\sqrt{M^2c^4 + {p_1^{(o)}}^2 c^2} + \sqrt{M^2c^4 + {p_2^{(o)}}^2 c^2}} = -\frac{\left(p_1^{(o)} + p_2^{(o)}\right) c}{E_1^{(o)} + E_2^{(o)}} \equiv -\frac{v^{(o)}}{c} \,,
\end{aligned} \tag{28}$$

where the last equation defines the velocity of the average reflector frame in O's inertial frame. Solving for $\tilde{\nu}_o/\nu_o$, we arrive at

$$\frac{\tilde{\nu}_o}{\nu_o} = \frac{1 + v^{(o)}/c}{1 - v^{(o)}/c} \,, \tag{29}$$

a formula that has the same algebraic structure as Eq. (8), as it should. Combining this with Eqs. (25), we obtain a result for the frequency shift observed by the emitter:

$$\begin{aligned}
\frac{\tilde{\nu}}{\nu} &= \sqrt{\frac{1 - w/c}{1 + w/c}}\,\tilde{\nu}_o \sqrt{\frac{1 - w/c}{1 + w/c}}\,\nu_o^{-1} = \frac{1 - w/c}{1 + w/c}\,\frac{\tilde{\nu}_o}{\nu_o} \,, \\
&\boxed{\frac{\tilde{\nu}}{\nu} = \frac{(1 - w/c)\left(1 + v^{(o)}/c\right)}{(1 + w/c)\left(1 - v^{(o)}/c\right)}}
\end{aligned} \tag{30}$$

Naturally, the question immediately arises, whether this is the same as (8) – which must of course be so, if everything is consistent. Let us check. $-v^{(o)}$ is the velocity of the average reflector frame in the inertial system of O, $-v$ is its velocity in the inertial frame of E, and w is the velocity of O in the frame of E. But these three velocities must satisfy the velocity addition theorem of special relativity, i.e., we must have

$$-v = \frac{w - v^{(o)}}{1 - wv^{(o)}/c^2} \,, \tag{31}$$

which allows us to express $v^{(o)}$ in terms of w and v:

$$\begin{aligned}
&-v + vwv^{(o)}/c^2 = w - v^{(o)} \quad \Rightarrow \quad v^{(o)}\left(1 + wv/c^2\right) = w + v \quad \Rightarrow \\
&v^{(o)} = \frac{w + v}{1 + wv/c^2} \,.
\end{aligned} \tag{32}$$

Plugging this into Eq. (30), we find

$$\begin{aligned}
\frac{\tilde{\nu}}{\nu} &= \frac{(1 - w/c)\left(1 + (w + v)/(1 + wv/c^2)/c\right)}{(1 + w/c)\left(1 - (w + v)/(1 + wv/c^2)/c\right)} = \frac{1 - w/c}{1 + w/c}\,\frac{1 + wv/c^2 + w/c + v/c}{1 + wv/c^2 - w/c - v/c} \\
&= \frac{1 - w/c}{1 + w/c}\,\frac{(1 + w/c)(1 + v/c)}{(1 - w/c)(1 - v/c)} = \frac{1 + v/c}{1 - v/c} \,, \quad \text{q.e.d.}
\end{aligned} \tag{33}$$

So the frequency shift observed by the emitter is independent of the velocity of the auxiliary observer frame (O), as it must, because our observer was not supposed to interfere in any way with the photon.

The discussion of the experiment from the point of view of an arbitrarily moving inertial frame of reference O makes the nature of the experiment very transparent. As we can see clearly now, the double Doppler frequency shift consists of three contributions in general:

1. a Doppler shift of the photon frequency between the emitter frame and the frame O ($\nu \to \nu_o$)
2. a frequency shift due to energy and momentum conservation during the collision process of the photon with the mirror ($\nu_o \to \tilde{\nu}_o$)
3. a Doppler shift of the returning photon's frequency between the frame O and the emitter frame ($\tilde{\nu}_o \to \nu_o$)

Thus, the double Doppler shift is by no means a pure Doppler effect, it is a combination of a collision-induced frequency shift with two Doppler shifts. The Doppler shifts do not follow from energy conservation, whereas the collision-induced shift does. The distribution of the full frequency shift between the two Doppler shifts and the collision-induced shift is frame dependent. The frequency ratio observed by the emitter is of course an objective quantity and frame independent.

Naturally, the two cases discussed before also fall under this general scheme, because what should prevent us from choosing, as our arbitrary inertial frame, either the emitter or the average reflector frame? Let us do so.

If the frame O is taken identical to E, we obviously have $w = 0$. This means that the two Doppler effects correspond to frequency shifts between non-moving frames, i.e. they are absent: $\nu_o = \nu$, $\tilde{\nu}_o = \tilde{\nu}$. The collision-induced frequency change is given by $\tilde{\nu}/\nu = (1 + v/c)/(1 - v/c)$, i.e., it corresponds to the full frequency shift seen by the emitter. Hence, the "double Doppler effect" does not include any Doppler shift in this case!

If the frame O is taken identical to R, the average reflector frame (as I have defined it), we have $w = -v$ and $v^{(o)} = 0$. This means that there is no frequency change in the frame O, the collision does not modify the frequency of the photon ($\tilde{\nu}' = \nu'$, but we never introduced $\tilde{\nu}'$). The total frequency change at the emitter is given by the concatenation of two Doppler shifts, each contributing a factor $\sqrt{(1 + v/c)/(1 - v/c)}$. So from the point of view of the average reflector frame, the double Doppler effect consists of two Doppler frequency shifts and nothing else. The energy balance during the collision does not contribute to the frequency shift.

Obviously, this is a much more differentiated view than the simplistic and categoric claim by Quattrini that the Doppler effect is physically due to energy conservation during an absorption process. He never showed this for the basic ("one-way") Doppler effect, which would be mandatory to support his claim. Instead, he discussed the complicated case of the double Doppler effect, where part of the frequency change may not even be due to a Doppler effect. But this escaped his attention, because he believed that this part constitutes the effect. However, as my discussion clarifies, it is just the remaining part (the two transitions between the frame in which the collision is described and the emitter frame) that is due to the Doppler effect. What he considers is the non-Doppler part of the frequency shift. (Which in the emitter frame is the whole shift, because if only one inertial frame is involved, there is no Doppler effect.)

Ironically, Quattrini pointed out a paper to me [6], in which the Doppler effect is derived for an atom emitting a photon and where the recoil energy appears in one form of the Doppler formula. Taking the mass of the atom to infinity seems to recover the standard form. However, the author demonstrates that this formula reduces to the standard form without approximation, via a correct identification of the photon frequency, and states: "One should be aware that the Doppler formula essentially gives the connection between energies of a single photon in two inertial frames. The mass of the source of radiation has nothing to do with the Doppler formula." This is quite the opposite of what Quattrini claims – he believes that the mass of the mirror (the source of radiation for the returning photon) must play an important role in an accurate description of the effect and that the infinite-mass case does not make any sense in a discussion of its cause. But the author of the paper he cited for me in order to support his point of view simply dismisses the claim as incorrect...

Two additional special frames

I would now like to discuss part of the description of the Doppler radar experiment in two further special frames. This will lead to additional physical insight and reward us with another neat formula for the double Doppler effect.

Before, let us rewrite our velocity result for the average reflector frame, defined in Eq. (28), in terms of velocities rather than momenta and energies. We have

$$p_i^{(o)} = M\gamma(v_i^{(o)})v_i^{(o)} , \quad i = 1,2 \qquad\qquad \text{where} \quad \gamma(v_i^{(o)}) = \frac{1}{\sqrt{1-(v_i^{(o)}/c)^2}} , \tag{34}$$

$$E_i^{(o)} = Mc^2\gamma(v_i^{(o)}) , \tag{35}$$

and we will often abbreviate

$$\beta_i = \frac{v_i}{c} , \qquad \gamma_i = \frac{1}{\sqrt{1-\beta_i^2}} \tag{36}$$

and indicate the velocity argument by either an accent, a subscript, and/or a superscript, for example $\beta_i^{(o)} = v_i^{(o)}/c$, $\gamma_i^{(o)} = \gamma(v_i^{(o)})$, $\hat{\beta}_i = \hat{v}_i/c$, $\check{\gamma}_i = 1/\sqrt{1-\check{\beta}_i^2}$, etc. The notation should be fairly easy to understand.

The rewriting yields:

$$\beta^{(o)} = \frac{v^{(o)}}{c} = \frac{\left(p_1^{(o)} + p_2^{(o)}\right)c}{E_1^{(o)} + E_2^{(o)}} = \frac{\left(M\gamma_1^{(o)}v_1^{(o)} + M\gamma_2^{(o)}v_2^{(o)}\right)c}{Mc^2\gamma_1^{(o)} + Mc^2\gamma_2^{(o)}} ,$$

$$\boxed{\beta^{(o)} = \frac{\beta_1^{(o)}\gamma_1^{(o)} + \beta_2^{(o)}\gamma_2^{(o)}}{\gamma_1^{(o)} + \gamma_2^{(o)}}} \tag{37}$$

The first frame we will consider is the *pre-collision* inertial frame of the mirror. We have assumed that before being hit by the photon the mirror moves at constant velocity $-v_1$ in the emitter frame. Now we define as our observer inertial frame the frame moving with the mirror velocity (i.e., $-v_1$) immediately before impact of the photon.[23] And we will indicate (most) quantities referring to that frame by a caret. So the velocity of the mirror in this frame just before impact of the photon is $\hat{v}_1 = 0$ (hence $\hat{\beta}_1 = 0$ and $\hat{\gamma}_1 = 1$) and its velocity[24]

[23] This way we even don't *have* to assume the mirror to be inertial, i.e. force-free, before arrival of the photon. Our inertial frame simply is the momentary inertial frame at rest with respect to the mirror just before impact of the photon, and it does not care about the velocities of the mirror some time before or (immediately) after impact of the photon. By defining this inertial system at one moment in time, we have defined it for all times, because we can of course calculate where any point of the system, whose velocity is fixed, had been prior to the time when its velocity and that of the mirror were equal, and we can calculate where any point of the inertial system will be in the far future, regardless of what the mirror is doing.

[24] Note that the v_i are defined as velocity components along the direction $\boldsymbol{e}_{x'}$, see Fig. 4, whereas in the emitter frame and in the arbitrary observer frame, velocity components are taken along the direction $\boldsymbol{e}_x = -\boldsymbol{e}_{x'}$, which leads to various minus signs in some formulas.

after reflection of the photon is $-\hat{v}_2$ with $\hat{p}_2 = M\hat{\gamma}_2\hat{v}_2$. Then we have, according to Eq. (37), $\hat{\beta} = \hat{\beta}_2\hat{\gamma}_2/(1+\hat{\gamma}_2)$ and obtain for the frequency change of the photon in this frame due to the collision, calling the frequency before collision ν_1 and that after collision $\tilde{\nu}_1$:

$$\begin{aligned}\frac{\tilde{\nu}_1}{\nu_1} &= \frac{1+\hat{\beta}}{1-\hat{\beta}} = \frac{1+\hat{\beta}_2\hat{\gamma}_2/(1+\hat{\gamma}_2)}{1-\hat{\beta}_2\hat{\gamma}_2/(1+\hat{\gamma}_2)} = \frac{1+\hat{\gamma}_2+\hat{\beta}_2\hat{\gamma}_2}{1+\hat{\gamma}_2-\hat{\beta}_2\hat{\gamma}_2} = \frac{1+\hat{\gamma}_2(1+\hat{\beta}_2)}{1+\hat{\gamma}_2(1-\hat{\beta}_2)} \\ &= \frac{1+\sqrt{\frac{1+\hat{\beta}_2}{1-\hat{\beta}_2}}}{1+\sqrt{\frac{1-\hat{\beta}_2}{1+\hat{\beta}_2}}} = \frac{\sqrt{\frac{1+\hat{\beta}_2}{1-\hat{\beta}_2}}+\frac{1+\hat{\beta}_2}{1-\hat{\beta}_2}}{\sqrt{\frac{1+\hat{\beta}_2}{1-\hat{\beta}_2}}+1} = \sqrt{\frac{1+\hat{\beta}_2}{1-\hat{\beta}_2}}\,. \end{aligned} \tag{38}$$

(This is just evaluating formula (29) for the current frame of reference.)

What we find is that the photon does undergo a (usually small) frequency change in the pre-collision inertial frame of the mirror (whereas it is reflected without frequency change in the average frame of the mirror). We may relate this to frequency changes in the emitter frame by using the basic Doppler formulas (without carets)

$$\nu_1 = \sqrt{\frac{1+\beta_1}{1-\beta_1}}\,\nu\,, \qquad \tilde{\nu} = \sqrt{\frac{1+\beta_1}{1-\beta_1}}\,\tilde{\nu}_1\,. \tag{39}$$

However, we will rather look at the second special frame now, which is the *after-collision* inertial frame of the mirror, moving at velocity $-v_2$ in the emitter frame. Indicating velocity-related quantities in that frame by an inverted caret, we obviously can write $\check{v}_2 = 0$ (hence $\check{\beta}_2 = 0$ and $\check{\gamma}_2 = 1$). The mirror velocity *before* reflection of the photon in the after-collision frame is $-\check{v}_1$ with $\check{p}_1 = M\check{\gamma}_1\check{v}_1$ and $\check{E}_1 = Mc^2\check{\gamma}_1$, and obviously we must have $\check{v}_1 = -\hat{v}_2$ due to the reciprocity of the relative velocities of the pre-collision and after-collision frames. This will become important later.

Moreover, we have $\check{\beta} = \check{\beta}_1\check{\gamma}_1/(1+\check{\gamma}_1)$ and obtain, calling the frequency of the photon before collision ν_2 and that after collision $\tilde{\nu}_2$,[25] from a calculation similar to (38):

$$\frac{\tilde{\nu}_2}{\nu_2} = \frac{1+\check{\beta}}{1-\check{\beta}} = \frac{1+\check{\beta}_1\check{\gamma}_1/(1+\check{\gamma}_1)}{1-\check{\beta}_1\check{\gamma}_1/(1+\check{\gamma}_1)} = \ldots = \sqrt{\frac{1+\check{\beta}_1}{1-\check{\beta}_1}}\,. \tag{40}$$

Again, it is clear that the photon does change its frequency due to the collision process in the after-collision frame (the change reduces to zero only for infinite mirror mass). We may relate this to frequency changes in the emitter frame by using the basic Doppler formulas (without inverted carets)

$$\nu_2 = \sqrt{\frac{1+\beta_2}{1-\beta_2}}\,\nu\,, \qquad \tilde{\nu} = \sqrt{\frac{1+\beta_2}{1-\beta_2}}\,\tilde{\nu}_2\,, \tag{41}$$

but first we study the relationship between the pre-collision and the after-collision frames a bit further.

The after-collision frame moves in the pre-collision frame at velocity $-\hat{v}_2$, so we may relate the photon frequencies in the two frames using the standard Doppler formula:

$$\nu_2 = \sqrt{\frac{1+\hat{\beta}_2}{1-\hat{\beta}_2}}\,\nu_1\,, \qquad \tilde{\nu}_1 = \sqrt{\frac{1+\hat{\beta}_2}{1-\hat{\beta}_2}}\,\tilde{\nu}_2\,. \tag{42}$$

[25] These are frequencies measurable in the after-collision frame.

Combining the second formula of Eq. (42) with Eq. (38), we find

$$\tilde{\nu}_2 = \sqrt{\frac{1-\hat{\beta}_2}{1+\hat{\beta}_2}}\,\tilde{\nu}_1 = \sqrt{\frac{1-\hat{\beta}_2}{1+\hat{\beta}_2}}\sqrt{\frac{1+\hat{\beta}_2}{1-\hat{\beta}_2}}\,\nu_1 = \nu_1\ , \tag{43}$$

i.e. the after-collision frequency of the photon in the after-collision frame is equal to the pre-collision frequency of the photon in the pre-collision frame! We also have, from the first formula of Eq. (42) and Eq. (40)

$$\nu_2 = \sqrt{\frac{1-\check{\beta}_1}{1+\check{\beta}_1}}\,\tilde{\nu}_2 = \sqrt{\frac{1-\check{\beta}_1}{1+\check{\beta}_1}}\sqrt{\frac{1-\hat{\beta}_2}{1+\hat{\beta}_2}}\,\tilde{\nu}_1 = \tilde{\nu}_1\ , \tag{44}$$

where the last equality holds, because $\check{\beta}_1 = -\hat{\beta}_2$. Hence, the photon changes frequency from ν_1 to $\tilde{\nu}_1$ in the pre-collision frame and it changes frequency from $\nu_2 = \tilde{\nu}_1$ to $\tilde{\nu}_2 = \nu_1$, i.e., by an equal amount in the opposite direction in the after-collision frame. This is very interesting...

Let us now calculate the frequency change in the emitter frame. Using (39) and (41) together with (43), we arrive at

$$\nu = \sqrt{\frac{1-\beta_1}{1+\beta_1}}\,\nu_1 = \sqrt{\frac{1-\beta_1}{1+\beta_1}}\,\tilde{\nu}_2 = \sqrt{\frac{1-\beta_1}{1+\beta_1}}\,\sqrt{\frac{1-\beta_2}{1+\beta_2}}\,\tilde{\nu}\ ,$$

$$\boxed{\frac{\tilde{\nu}}{\nu} = \sqrt{\frac{1+\beta_1}{1-\beta_1}}\,\sqrt{\frac{1+\beta_2}{1-\beta_2}}} \tag{45}$$

This is a new formula for the double Doppler effect. It must be equivalent to (8) (together with (7)), but this is not so straightforward to see. I will demonstrate the equivalence in the next section.

Quattrini obtains this formula by a simpler calculation. My approach has the advantage of making the physics more transparent. The crucial point to observe is that we have now used not two but three frames of reference and that this in principle leads to four frequency shifts instead of three before. Three of these are Doppler shifts, rather than two.

The three frames are, of course, the emitter frame, the pre-collision frame and the after-collision frame. There is a Doppler shift from the frequency ν to ν_1, involving a factor of $((1+\beta_1)/(1-\beta_1))^{1/2}$, on switching from the emitter frame to the pre-collision frame. Then there is a frequency shift from ν_1 to $\tilde{\nu}_1$, involving a factor $((1+\hat{\beta}_2)(1-\hat{\beta}_2))^{1/2}$ due to the collision of the photon with the mirror.[26] Switching now to the after-collision frame in the description, the frequency of the photon changes again, due to the Doppler effect, now from $\tilde{\nu}_1$ to $\tilde{\nu}_2$, which produces a factor $((1+\check{\beta}_2)(1-\check{\beta}_2))^{1/2} = ((1-\hat{\beta}_2)(1+\hat{\beta}_2))^{1/2}$ that *cancels* the collision-induced shift, i.e., the frequency of the photon now is back to ν_1 (because $\tilde{\nu}_2 = \nu_1$).[27] Finally, there is a Doppler shift between the after-collision frame and the emitter frame, from frequency $\tilde{\nu}_2 = \nu_1$ to $\tilde{\nu}$, and this involves a factor of $((1+\beta_2)/(1-\beta_2))^{1/2}$. Since the two frequency shifts of the middle part of the process cancel each other, all that remains are the two factors $((1+\beta_1)/(1-\beta_1))^{1/2}$ and $((1+\beta_2)/(1-\beta_2))^{1/2}$, giving rise to (45), seemingly just a double Doppler effect.

Formula (45) alone is insufficient to calculate a velocity from a measurement of the frequency ratio $\tilde{\nu}/\nu$, which gives one parameter of the experiment only, because there are two velocities

[26] Note that $\hat{\beta}_2$ is negative, so this factor reduces the frequency.

[27] Steps 2 and 3 can be interchanged, i.e., we may transform the photon frequency ν_1 directly into ν_2, using the Doppler shift factor $((1+\hat{\beta}_2)(1-\hat{\beta}_2))^{1/2}$, and then consider reflection in the after-collision frame, which shifts the frequency to $\tilde{\nu}_2$, i.e., back to ν_1.

on the right-hand side, which cannot both be evaluated from a single parameter without further information. So the formula is less useful in direct measurements than the result (8), which always gives one a velocity result, even if that velocity will not exactly agree with either v_1 or v_2, unless the mass of the mirror is infinitely large. However, formula (45) could be very useful in the theoretical description of the "Quattrini" experiment where we have many bounces back and forth by the photon, if we wish to take into account the fact that the emitter itself also does not have infinite mass and therefore suffers a recoil. Using (45) instead of (8) in the center-of-energy frame of the system would allow one to work with the actual velocities of both the emitter and reflector before and after bounces, allowing one to avoid the introduction of frames with average velocities. It would then be possible, in the calculation of a sequence of bounces to use the calculated velocity changes on each bounce directly (as the frames considered would always be momentary pre-collision or after-collision frames), instead of first calculating a frame moving at an average velocity between the pre- and after-collision velocities and then use that to get back to the actual velocities themselves, in order to prepare the calculation for the next bounce. I will however not pursue the option of dealing with a movable emitter. That would lead us too far astray.

Equivalence of several formulas for the double Doppler effect

My result (8), rewritten in terms of β and γ factors, reads:

$$\frac{\tilde{\nu}}{\nu} = \frac{1+\beta}{1-\beta}\,. \qquad \text{where} \quad \beta = \frac{\beta_1\gamma_1 + \beta_2\gamma_2}{\gamma_1+\gamma_2}\,. \tag{46}$$

I first will show that this is the same as Eq. (45). There are some useful relationships involving the β's and γ's allowing one to recast expressions without producing too many square roots. We note

$$\begin{aligned}\beta_1^2\gamma_1^2 - \beta_2^2\gamma_2^2 &= \frac{\beta_1^2}{1-\beta_1^2} - \frac{\beta_2^2}{1-\beta_2^2} = 1 + \frac{\beta_1^2}{1-\beta_1^2} - 1 - \frac{\beta_2^2}{1-\beta_2^2}\\ &= \frac{1}{1-\beta_1^2} - \frac{1}{1-\beta_2^2} = \gamma_1^2 - \gamma_2^2\end{aligned} \tag{47}$$

and both the initial and final expressions can be factorized with the help of a binomial expression:

$$\begin{aligned}&(\beta_1\gamma_1 - \beta_2\gamma_2)(\beta_1\gamma_1 + \beta_2\gamma_2) = (\gamma_1-\gamma_2)(\gamma_1+\gamma_2)\\ &\Rightarrow \quad \frac{\beta_1\gamma_1+\beta_2\gamma_2}{\gamma_1+\gamma_2} = \frac{\gamma_1-\gamma_2}{\beta_1\gamma_1-\beta_2\gamma_2}\,.\end{aligned} \tag{48}$$

The second equation is valid, if we do not divide by zero, i.e., if the two velocities are different. Its left-hand side is just the definition of β in (46). So let us use this:

$$\begin{aligned}\frac{\tilde{\nu}}{\nu} &= \frac{1+(\gamma_1-\gamma_2)/(\beta_1\gamma_1-\beta_2\gamma_2)}{1-(\gamma_1-\gamma_2)/(\beta_1\gamma_1-\beta_2\gamma_2)} = \frac{\beta_1\gamma_1-\beta_2\gamma_2+\gamma_1-\gamma_2}{\beta_1\gamma_1-\beta_2\gamma_2-\gamma_1+\gamma_2} = \frac{\gamma_1(1+\beta_1)-\gamma_2(1+\beta_2)}{-\gamma_1(1-\beta_1)+\gamma_2(1-\beta_2)}\\ &= \frac{(1+\beta_1)/\gamma_2-(1+\beta_2)/\gamma_1}{-(1-\beta_1)/\gamma_2+(1-\beta_2)/\gamma_1} = \frac{(1+\beta_1)\sqrt{1-\beta_2^2}-(1+\beta_2)\sqrt{1-\beta_1^2}}{-(1-\beta_1)\sqrt{1-\beta_2^2}+(1-\beta_2)\sqrt{1-\beta_1^2}}\\ &= \frac{\sqrt{1+\beta_1}\sqrt{1+\beta_2}(\sqrt{1+\beta_1}\sqrt{1-\beta_2}-\sqrt{1-\beta_1}\sqrt{1+\beta_2})}{\sqrt{1-\beta_1}\sqrt{1-\beta_2}(-\sqrt{1-\beta_1}\sqrt{1+\beta_2}+\sqrt{1+\beta_1}\sqrt{1-\beta_2})}\,.\end{aligned} \tag{49}$$

Close inspection shows that the expressions in parentheses in the numerator and denominator are the same (the second term in the denominator expression is equal to the first of the numerator and the first term in the denominator corresponds to the second in the numerator). So these whole long parentheses cancel and we are left with

$$\frac{\tilde{\nu}}{\nu} = \sqrt{\frac{1+\beta_1}{1-\beta_1}}\sqrt{\frac{1+\beta_2}{1-\beta_2}}\,, \qquad \text{q.e.d.} \tag{50}$$

This demonstrates that both formulas produce the exact frequency shift for the double Doppler effect, regardless of the recoil of the mirror. Neither of them alone is sufficient to calculate the pre-collision and after-collision velocities of the mirror. The first formula, however, gives us the velocity (relative to the emitter frame) of an inertial frame of reference, in which the photon is reflected without energy change, and this velocity agrees with the velocity of the mirror for large mirror masses. In the latter case, the second formula can, of course, *also* be used for the evaluation of the mirror velocity by setting $v_1 = v_2$.

If the mass of the mirror is known or has been measured, formulas for both v_1 and v_2 can be developed using either double Doppler result. The easier approach seems to proceed via (50). To obtain the additional information needed to calculate two quantities, we go back to our initial energy and momentum equations (1,2), in order to obtain expressions for ν and $\tilde{\nu}$ in addition to their ratio.

$$h\nu + Mc^2\gamma_1 = h\tilde{\nu} + Mc^2\gamma_2\,, \tag{51}$$

$$h\nu - Mc^2\beta_1\gamma_1 = -h\tilde{\nu} - Mc^2\beta_2\gamma_2\,, \tag{52}$$

which can be compactly recast as

$$\frac{h(\nu-\tilde{\nu})}{Mc^2} = \gamma_2 - \gamma_1\,, \tag{53}$$

$$\frac{h(\nu+\tilde{\nu})}{Mc^2} = \beta_1\gamma_1 - \beta_2\gamma_2\,. \tag{54}$$

Adding and subtracting the two equations, we get expressions for ν and $\tilde{\nu}$

$$2\frac{h\nu}{Mc^2} = -(1-\beta_1)\gamma_1 + (1-\beta_2)\gamma_2\,, \tag{55}$$

$$2\frac{h\tilde{\nu}}{Mc^2} = (1+\beta_1)\gamma_1 - (1+\beta_2)\gamma_2\,. \tag{56}$$

(Taking the ratio of (53) and (54), we can directly derive (46) and/or (50).) Noting that we may rewrite Eq. (50) as

$$\frac{\tilde{\nu}}{\nu} = (1+\beta_1)\gamma_1\,(1+\beta_2)\gamma_2 = \frac{1}{(1-\beta_1)\gamma_1\,(1-\beta_2)\gamma_2}\,, \tag{57}$$

the idea is then to use either (55) or (56) to express one of the β-γ expressions by the other and a mass term and to substitute this into (57) to obtain a result containing only one of the velocities and the mass. We use Eq. (55), because that will produce ν on the right-hand side rather than $\tilde{\nu}$. ν is usually given in the experiment (it is the frequency that the emitter creates), whereas $\tilde{\nu}$ is to be determined by measurement.

We first isolate the terms dependent on the after-collision velocity ($-v_2$):

$$(1-\beta_2)\gamma_2 = (1-\beta_1)\gamma_1 + 2\frac{h\nu}{Mc^2}\,, \tag{58}$$

and plug this into (57) to get

$$\frac{\tilde{\nu}}{\nu} = \frac{1}{(1-\beta_1)\gamma_1((1-\beta_1)\gamma_1 + 2h\nu/(Mc^2))} = \frac{1}{(1-\beta_1)^2\gamma_1^2 + (1-\beta_1)\gamma_1 2h\nu/(Mc^2)}$$

$$= \frac{1}{(1-\beta_1)^2/(1-\beta_1^2) + \sqrt{(1-\beta_1)/(1+\beta_1)}2h\nu/(Mc^2)} ,$$

$$\frac{\tilde{\nu}}{\nu} = \frac{1+\beta_1}{1-\beta_1+2h\nu/(\gamma_1 Mc^2)} , \tag{59}$$

which is the desired result and also an exact one, expressing the frequency ratio via terms containing only the velocity v_1 (related to the pre-collision state of the mirror) and the mass. The right-hand side probably cannot be solved analytically for the velocity, except in the case $\gamma_1 \approx 1$, but it would be no problem to evaluate the velocity numerically, once the measurement of $\tilde{\nu}$ and the mass M has been made. Note also that the term $h\nu/(\gamma_1 Mc^2)$ is the ratio of the photon energy to the pre-collision energy of the mirror. The recoil energy does not appear in the formula. (It is defined as the difference between the kinetic energies of the mirror after and before the interaction with the photon, so here it would take the value $Mc^2(\gamma_2 - \gamma_1)$. Since it depends on both velocities of the mirror, its appearance in a formula would render it more difficult to use the formula for the evaluation of only one of the velocities.)

Finally, let us develop a formula for the after-collision velocity $(-v_2)$ of the mirror. Normally that would be more interesting than the pre-collision velocity, because we would like to know what velocity the mirror has after measurement of the Doppler shift, not what velocity it had before. It does not have this velocity anymore after the experiment, so our measurement would be less useful for further predictions of the mirror behavior.

We recast (55) as

$$(1-\beta_1)\gamma_1 = (1-\beta_2)\gamma_2 - 2\frac{h\nu}{Mc^2} , \tag{60}$$

and obtain from (57)

$$\frac{\tilde{\nu}}{\nu} = \frac{1}{(1-\beta_2)\gamma_2((1-\beta_2)\gamma_2 - 2h\nu/(Mc^2))} = \frac{1}{(1-\beta_2)^2\gamma_2^2 - (1-\beta_2)\gamma_2 2h\nu/(Mc^2)} ,$$

$$\frac{\tilde{\nu}}{\nu} = \frac{1+\beta_2}{1-\beta_2-2h\nu/(\gamma_2 Mc^2)} , \tag{61}$$

which is again an exact result, now allowing us to express the frequency shift by the after-collision velocity of the mirror and its mass (plus the frequency of the incoming photon).

Of course, we have

$$\frac{\tilde{\nu}}{\nu} = \frac{1+\beta}{1-\beta} = \sqrt{\frac{1+\beta_1}{1-\beta_1}}\sqrt{\frac{1+\beta_2}{1-\beta_2}} = \frac{1+\beta_1}{1-\beta_1+2h\nu/(\gamma_1 Mc^2)} = \frac{1+\beta_2}{1-\beta_2-2h\nu/(\gamma_2 Mc^2)} , \tag{62}$$

as no approximations were used during the transformation of one of these formulas into another.

Conclusions

Let me briefly summarize what has been achieved in this essay.

- An exact result for the double Doppler effect in the case of finite reflector mass, i.e., in the presence of recoil, has been derived. The treatment of the reflection process was done in a large variety of inertial systems,[28] using energy and momentum conservation in the same form in each of them, meaning that the equivalence of inertial systems (i.e., the relativity principle) was applied. Obviously this disproves Quattrini's claim of inapplicability of the equivalence of inertial systems.

- The derived formula is formally identical to the infinite-mass result and agrees with what is predicted by a succession of two Lorentz transformations, thus disproving Quattrini's claim to the contrary. All that is different with respect to the infinite-mass case is that the velocity of the inertial system to which the formula refers is not identical with the mirror velocity, because the mirror does not stay at rest in the same inertial system during the collision with the photon, unless its mass is infinite. Rather the velocity of the inertial system is a well-defined average of the pre-collision and after-collision velocities of the mirror.
- It was clearly pointed out that inertial systems are not attached to the masses the motion of which may have served for their original definition. Whenever a force is exerted on the mass, it accelerates and loses its inertial state of motion. That does of course not destroy the inertial system, in which the mass originally was at rest. All that happens is that the mass now moves in this inertial system instead of being at rest.
- The formula derived was demonstrated to be mathematically equivalent to various formulas for the double Doppler effect given by Quattrini, who wanted to show that my formula could only be an approximation. Since Quattrini's formulas were not approximations, mine is not an approximation either.

The picture that emerges of the double Doppler effect, from these derivations, is that it is composed of three phenomena, in general, only one of which is dynamical. That is the frequency shift of the photon due to the collision with the mirror. Frequencies and energies are frame dependent, so that frequency shift is different in different inertial systems, in general. The other two phenomena are the two Doppler shifts relating the photon frequencies (before and after the encounter with the mirror) in the inertial system under consideration to the frequencies observed by the emitter, i.e., the observer measuring the double Doppler effect. These Doppler shifts have nothing to do with energy conservation. The energy conserving collision process, on the other hand, does not produce a Doppler frequency shift. The definition of the Doppler effect says it is a frequency change between two observers moving with respect to each other; this is not the case for the frequency shift in the collision process which is a shift for a single observer.

This distinction is corroborated by a three-frame description of the effect, in which we use, besides the emitter frame, the pre-collision and after-collision frames of the mirror (both inertial), one to describe the collision-induced frequency change and the other to transfer it to a frame that is more convenient for the description of the returning photon. Here, it turns out that the Doppler effect between the frames cancels the collision-induced frequency change, emphasizing that these are two different types of frequency modification and giving a physical interpretation to Quattrini's two-velocity double Doppler formula.

Finally, it may be noted that here we have used the quantum mechanical relation between energy and frequency to describe electromagnetic waves as photons, i.e., as particles, and the interaction with the mirror as a collision process. Of course, the double Doppler effect, not being a quantum mechanical effect, can also be derived in purely classical electrodynamics, without any use of the energy-frequency relationship. What would have to be done is the usual description of the electromagnetic problem via an ansatz for the incoming, transmitted and reflected waves and a setup of the boundary conditions for these waves on the mirror

[28] Essentially, in all inertial systems moving parallel to the photon wave vector.

surface, for a moving mirror instead of, as usual, for a fixed one. Poynting's theorem could be used for a calculation of the momentum transfer between the field and the mirror, but energy conservation would not arise directly in the approach. Instead, the matching conditions for the three waves[29] at the mirror surface, which lead to the requirement of equal frequencies for a mirror at rest, would lead to different frequencies for the incoming and reflected waves, due to the Doppler effect. (So the description in classical electrodynamics would seem to always require at least two frames, including the frame of the moving mirror surface.)

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