Limiting behaviour of a class of analytic functions for large arguments on the real axis

KLAUS KASSNER 2 December 2016

Because writing a lot of formulas into the Research Gate answer window is awkward, I type this up in LaTeX. Also this gives me the opportunity to do a fairly complete summary.

The set of functions I want to consider is described by

$$f_k(x) = \sin x \, \mathrm{e}^{-x^{2k} \sin^2 x} , \qquad k \in \mathbb{N},\, k > 0 . \tag{1}$$

Clearly, all of these functions are analytic in the whole complex plane. Since they are neither polynomials nor constants, this means they must have an essential singularity at the point $x = \infty$.

In particular, I have looked, in preceding discussions, at the two most interesting representatives[1] of the set,

$$f_1(x) = \sin x \, \mathrm{e}^{-x^2 \sin^2 x} \tag{2}$$

and

$$f_2(x) = \sin x \, \mathrm{e}^{-x^4 \sin^2 x} . \tag{3}$$

What I'd like to discuss in some detail today is which of the three assertions

(a) $\lim_{x\to\infty} f(x) = 0$,

(b) $\lim_{x\to\infty} f'(x)$ does not exist,

(c) $f(x)$ is of bounded variation on the interval $[x_0, \infty)$, where $x_0 \in \mathbb{R}$ is arbitrary (but finite)

are satisfied for which values of k by functions from the set.

Let us first calculate their derivatives:

$$f_k'(x) = \left(\cos x - 2k x^{2k-1} \sin^3 x - 2x^{2k} \sin^2 x \cos x \right) \mathrm{e}^{-x^{2k} \sin^2 x} . \tag{4}$$

We can then immediately state that *assertion (b) is satisfied by $f_k(x)$ for all allowed values of k.* This follows from

$$f_k'(n\pi) = (-1)^n , \qquad \text{for } n \in \mathbb{N} , \tag{5}$$

which is due to the fact that $\sin(n\pi) = 0$, pushing to zero all the terms in the parentheses except for the cosine one $[\cos(n\pi) = (-1)^n]$ and setting the exponential to one.

Next, we consider assertion (c). [A statement on (a) will follow from the result.] Since $f'(x)$ is continuous, we may calculate the variation on any interval $[a, b]$ from

$$V_a^b(f) = \int_a^b \left| f'(x) \right| \mathrm{d}x . \tag{6}$$

[1] If we were willing to study a generalization, not requiring analyticity, we could replace k by some real exponent β, and the two interesting subsets would then be the functions with $\beta \leq 1$ and those with $\beta > 1$.

On each interval, on which $f(x)$ is monotonous, this gives $V_a^b(f) = |f(b) - f(a)|$, so in order to do the calculation, it is useful to know the positions of the extrema of $f(x)$, because $f(x)$ is monotonous between two successive isolated extrema.[2] That is, we should find the zeros of $f'(x)$. This entails solving the transcendental equation

$$\cos x - 2kx^{2k-1}\sin^3 x - 2x^{2k}\sin^2 x\cos x = 0\,. \tag{7}$$

Because it is difficult (or impossible) to obtain a general analytic solution, we restrict ourselves to the case $x \gg 1$. We may do so, because it is possible to divide our interval $[x_0, \infty)$ into two subintervals $[x_0, x_1]$ and $[x_1, \infty)$ with $x_1 \gg 1$, and we know that $f(x)$ is of bounded variation on $[x_0, x_1]$, because the integral (6) is definitely finite for $a = x_0$, $b = x_1$, x_0, x_1 finite, due to the boundedness of its – continuous – integrand. Thus, in order to assess whether the total variation is bounded or not, it is sufficient to consider the variation on the interval $[x_1, \infty)$ with $x_1 \gg 1$. So we will look for solutions of (7) for $x \gg 1$ only and we will assume, for definiteness, $n\pi < x < (n+1)\pi$, i.e. restrict attention to an interval on which $f(x) \neq 0$. Clearly, this is not a restriction of generality. We rewrite (7) as

$$kx^{2k-1}\tan x\sin^2 x + x^{2k}\sin^2 x = \frac{1}{2}\,, \tag{8}$$

and solve it using the method of dominant balance. Now when I first looked at this for $k = 1$, I immediately saw the dominant balance

$$k\tan x \sim -x \qquad (x \gg 1)\,. \tag{9}$$

Setting $x = \pi\left(n + \frac{1}{2}\right) + \delta$, we see that the requirement for the tangent to become large leads to $\tan x \approx -\frac{1}{\delta}$, whence $\delta \approx \frac{k}{\pi(n+1/2)}$ (and $\cos x$ must be small, hence $\sin x \approx 1$). The terms considered in the balance were

$$kx^{2k-1}\tan x\sin^2 x \sim -x^{2k}\sin^2 x \sim -\pi^{2k}\left(n + \frac{1}{2}\right)^{2k} \gg \frac{1}{2}\,, \tag{10}$$

so the dominant balance is indeed consistent.[3]

At the extrema described by this, we have

$$f_k(x) \approx \mathrm{e}^{-x^{2k}\sin^2 x} \approx \mathrm{e}^{-x^{2k}} \approx \mathrm{e}^{-[\pi(n+1/2)]^{2k}}\,. \tag{11}$$

Obviously, if this was the only extremum in the interval $n\pi < x < (n+1)\pi$, then it would have to be a local maximum of the absolute value of $f(x)$ on the interval, because $f(x)$ is zero at its boundaries. The variation on the interval would then be smaller than twice the value of this maximum, and since the sequence decays exponentially fast (even in n^2) the total variation of f_k would be finite for all $k \in \mathbb{N}\backslash 0$. This was, in fact, what I first believed. But that was too rash. Care requires to check whether (7) has additional solutions in the interval $[n\pi, (n+1)\pi]$. Since there are only three terms, there are only three pairs of possible dominant balances, of which we have already found one.

[2] In cases, where $f(x)$ has constant pieces on which it is extremal (which is forbidden for the $f_k(x)$ by their analyticity), we take just one point from the interval, on which it is constant as a representative of the sequence of extrema. All that matters is that $f(x)$ is monotonous on the subintervals considered.

[3] In asymptotology, if $x \gg 1$ $(x \to \infty)$ then $-x \gg 1$ $(x \to \infty)$, too, because $g(x) \gg h(x)$ $(x \to \infty)$ simply means $\lim_{x\to\infty} h(x)/g(x) = 0$. In the example, we stay near the center of the interval $[\pi n, \pi(n+1)]$ and just take n large, so the sine cannot have a zero.

Another possibility would be

$$kx^{2k-1}\tan x\sin^2 x \sim \frac{1}{2} \quad (x \gg 1) \qquad \Rightarrow \qquad \sin^3 x \sim \frac{\cos x}{2kx^{2k-1}}\,. \tag{12}$$

For large x, this is satisfiable only, if $\sin x$ is small, which is the case near the ends of the interval. Consider $x = n\pi + \delta$ for definiteness with $\delta > 0$, $\delta \ll 1$. Then $\cos x \approx (-1)^n$ and $\delta^3 \approx (-1)^n/2k(n\pi)^{2k-1}$, but then the term that we neglected in our dominant balance,

$$x^{2k}\sin^2 x \approx x^{2k}\left(\frac{|\cos x|}{2kx^{2k-1}}\right)^{2/3} \approx \left(\frac{1}{2k}\right)^{2/3} x^{(2k+2)/3} \gg 1\,, \tag{13}$$

which shows that neglecting it in comparison with $\frac{1}{2}$ was unjustified. So this pairing does not lead to a dominant balance nor to new extrema. And of course that holds for the case $x = (n+1)\pi - \delta$ as well.

Therefore, let us look at the final possibility for a balance between two terms of (8)

$$x^{2k}\sin^2 x \sim \frac{1}{2} \qquad \Rightarrow \qquad |\sin x| \sim \frac{1}{\sqrt{2}x^k}\,. \tag{14}$$

This is solved, in the interval $[n\pi, (n+1)\pi]$, by $x = n\pi + \delta$ with $\delta \approx 1/\left[\sqrt{2}(n\pi)^k\right]$ and by $x = (n+1)\pi - \delta$ with $\delta \approx 1/\left[\sqrt{2}((n+1)\pi)^k\right]$. Moreover, we can check ($|\cos x| \approx 1$)

$$\left|kx^{2k-1}\tan x\sin^2 x\right| \approx kx^{2k-1}|\sin x|\,\frac{1}{2x^k} \approx \frac{k}{2}\frac{|\sin x|}{x} \ll \frac{1}{2}\,, \tag{15}$$

that is, neglect of the tangent term is justified, we have found another dominant balance. Estimating the magnitude of $f_k(x)$ at the corresponding x values, we find

$$|f_k(x)| \approx \frac{1}{\sqrt{2}x^k}\mathrm{e}^{-x^{2k}/2x^{2k}} = \frac{1}{\sqrt{2}x^k}\mathrm{e}^{-1/2}\,. \tag{16}$$

This is much larger than the function values at the first dominant balance. We therefore conclude, that $f_k(x)$ has, at least for large enough x, three extrema between any pair of zeros $x = n\pi$ and $x = (n+1)\pi$. The extremum for x just above $n\pi$ is a maximum for n even ($\sin x$ positive in the interval), then the extremum near $x = (n+\frac{1}{2})\pi$ must be a minimum, and the one just below $(n+1)\pi$ a maximum again. In intervals with n odd, we have the sequence minimum, maximum, minimum. As to the absolute value of $f_k(x)$, that is always maximum near the interval boundaries and minimum near its middle.

Because the absolute value of the maxima of $f_k(x)$ in closed intervals of length π goes to zero as $1/n^k$ for $n \to \infty$, we may immediately conclude that $\lim_{x\to\infty} f_k(x) = 0$ from the continuity of $f_k(x)$. This conclusion is independent of the value of k, as long as it is bigger than zero. *Therefore, assertion (a) is true for all the $f_k(x)$.*

Moreover, given that there are two maxima of the absolute value of $f_k(x)$ in the interval $[n\pi, (n+1)\pi]$, we have the estimate, if $x_{n,\max}$ is the x coordinate of the larger of the two:

$$|f_k(x_{n,\max})| < V_{n\pi}^{(n+1)\pi}(f_k) < 4\,|f_k(x_{n,\max})|\,.$$

(f_k varies from zero to the maximum at least once, but its total variation on the interval can be no more than four times that variation, because there is just one other local maximum in the interval and it is not as large as the global one.)

If we then take $x_1 = n_1\pi$, we may estimate:

$$
\begin{aligned}
V_{x_1}^{\infty}(f_1) &> \sum_{m=n_1}^{\infty} \frac{1}{\sqrt{2}x_{m\max}}\,\mathrm{e}^{-1/2} > \sum_{m=n_1}^{\infty} \frac{1}{\sqrt{2}(m+1)\pi}\,\mathrm{e}^{-1/2} = \infty\,, \\
V_{x_1}^{\infty}(f_k) &< \sum_{m=n_1}^{\infty} \frac{4}{\sqrt{2}x_{m\max}^{k}}\,\mathrm{e}^{-1/2} < \sum_{m=n_1}^{\infty} \frac{2\sqrt{2}}{m^k\pi^k}\,\mathrm{e}^{-1/2} < \infty\,, \qquad \text{for } k \geq 2\,.
\end{aligned} \tag{17}
$$

Since the total variation of f_k on the interval $[x_0, \infty)$ is obtained by adding a finite quantity to the variation on $[x_1, \infty)$, we can state that *assertion (c) is not satisfied for* $k = 1$, *but is true for all* $k \geq 2$. Of course, the antisymmetry of $f_k(x)$ allows an immediate extension to the complete interval $(-\infty, \infty)$. We can say that $f_1(x)$ is not of bounded variation on $(-\infty, \infty)$ and that $f_2(x)$ as well as all $f_k(x)$ with $k \in \mathbb{N}$, $k > 2$ are of bounded variation on $(-\infty, \infty)$.

Finally, in the context of the ResearchGate discussion, it may be interesting to consider the asymptotics of $f_k(x)$ for $x \to \infty$ along the real axis. To this end, it is useful to note that $f_k(x)$ satisfies the following differential equation:

$$
f_k'(x) = \Big(\cot x - 2kx^{2k-1}\sin^2 x - 2x^{2k}\sin x\cos x\Big)\, f_k(x)\,. \tag{18}
$$

A local asymptotic analysis of this about the irregular singular point $x = \infty$ gives, as the leading behaviour of $f_k(x)$, the exact solution of the differential equation. Hence the asymptotic series for $f_k(x)$ is given (in standard asymptotic analysis) by the function itself and contains only one term, i.e. it terminates after the exponential prefactor. The controlling factor is $\mathrm{e}^{-x^{2k}\sin^2 x}$, which is not of bounded variation, whereas the leading behaviour includes the sine prefactor and is of bounded variation for $k > 1$. Note that the product of two factors which are not of bounded variation produces an expression of bounded variation here.

What this little study shows is that the following claim, which certainly was the interpretation by some readers of a number of posts in this thread, "If $\lim_{x\to\infty} f(x) = 0$ and $f(x)$ is of bounded variation on $[x_0, \infty)$, then $\lim_{x\to\infty} f'(x) = 0$, too." is *wrong*. But at least this claim would have had the form of a serious mathematical proposition. Since the author of the posts, which were read as claiming this, states what he actually meant to say was something like "If $\lim_{x\to\infty} f(x) = 0$ and the expression of $f(x)$ contains only functions of bounded variation on $[x_0, \infty)$, then $\lim_{x\to\infty} f'(x) = 0$, too.", let us consider this variant briefly.

The first thing to note is that the statement is pretty ambiguous. Mathematics provides us with many ways to express functions, so "the expression of $f(x)$" is not a unique concept. There may be ways to express a function of bounded variation with and without the intermediate use of functions of unbounded variation. Would such a function qualify as one the expression of which contains only functions of bounded variation? Morever, most functions of x will have x in their expression, and this, i.e., the identity function, is not of bounded variation on the interval $[x_0, \infty)$. If we had to restrict ourselves to the functions not containing x in their definition, we could consider constants only. For these, the statement is true, but very trivial. The only *constant* function having the limit 0 at infinity is $f(x) \equiv 0$, and this certainly satisfies $\lim_{x\to\infty} f'(x) = 0$ as well. Now the author of the proposition says this was not what he meant and gives $\sin x$ or $\sin x^2$ as the functions he wanted to exclude. However, examples don't exhaust a definition. *One* counterexample is sufficient to show a theorem to be wrong, but a definition must allow us to decide for *each* case whether it a applies to it or not. So let me try to give a definition that excludes $\sin x^2$ but not x. This would read "a function that is bounded but of not of bounded variation on $[x_0, \infty)$". Is it true that if we exclude this type of functions from the "expression of $f(x)$" that $\lim_{x\to\infty} f(x) = 0$ implies $\lim_{x\to\infty} f'(x) = 0$? I don't think this helps. We may write $\sin x^2 = \pi x^2 \prod_{n=1}^{\infty} \left(1 - \frac{x^4}{n^2}\right)$ in any place where it appears and would have rewritten the expression of our function in a form that does not contain any bounded functions of unbounded variation,[4] which is supposedly sufficient to guarantee the limit for $x \to \infty$ of $f'(x)$ to be equal to zero, if the limit of $f(x)$ is. However, $f_2(x)$ given above works as a perfect counterexample for this proposition, if we rewrite the sine as an infinite product or a series.

My point is that a statement such as "the expression of $f(x)$ does not contain any functions of unbounded variation" is meaningless without precise rules indicating what are legitimate or "well-formed" expressions. Once we have these, it might be possible to discuss whether there is a non-trivial sense in which a proposition referring to bounded variation of terms in the expression of $f(x)$ may be useful to decide the question about the limit of $f'(x)$.

On the other hand, such a proposition may not be necessary, given that we have very simple conditions on the behaviour of the function $f'(x)$ that, together with assertion (a) ensure that $\lim_{x\to\infty} f'(x) = 0$. Either of the conditions that $\lim_{x\to\infty} f'(x)$ exists or that $f'(x)$ [instead of $f(x)$] is of bounded variation is sufficient that $\lim_{x\to\infty} f(x) = 0$ implies $\lim_{x\to\infty} f'(x) = 0$, as I have discussed elsewhere.

[4]The series expansion of the sine would do as well. I use the product representation just to show that one and the same function may be expressed in very different, possibly exotic ways, and that therefore a proposition referring to the "expression of a function" is problematic.